Institut de Radiophysique

Cox-Isham-Smith β-γ coincidence rate correction

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Introduction

- Many of us inherit an infrastructure for coincidence counting, which involves the imposition of non-extending and/or extending deadtimes.

- β-γ coincidence counting rates, when using non-extending deadtimes (NEDT), require corrections.

- Historically, many corrections were proposed, including those of Campion (1959), Gandy (1961,1962), Hayward (1961), Bryant (1963), and later Cox-Isham (1977) and Cox-Isham-Smith (1978).

Introduction

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Printed in Great Britain

A bivariate point process connected with electronic counters

By D. R. Cox, F.R.S. and Valerie Isham
Department of Mathematics, Imperial College, London

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IMPROVED CORRECTION FORMULAE FOR COINCIDENCE COUNTING

DAVID SMITH
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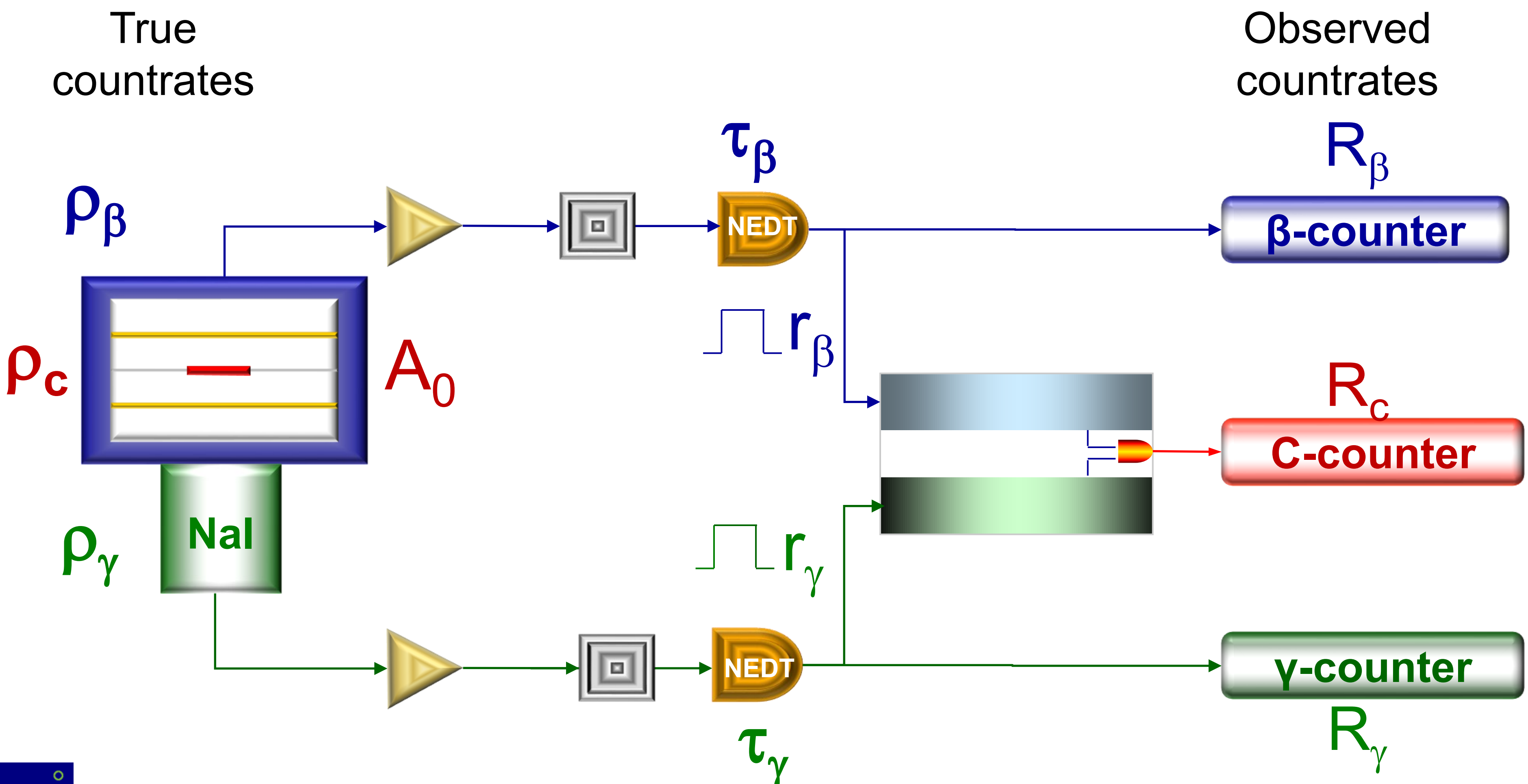
Introduction

- Here we discuss the Cox-Isham-Smith (CIS) correction.

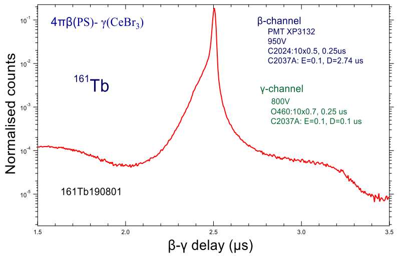
- CIS correction is exact for $\tau_\beta = n \cdot \tau_\gamma$ or $\tau_\gamma = n \cdot \tau_\beta$, n integer.

- CIS correction is highly accurate for $\tau_\beta = s \cdot \tau_\gamma$, s non-integ.
  - ➡ 4th order in $\tau_{\beta,\gamma}$ and 3rd order in $r_{\beta,\gamma}$;
  - ➡ Allows for β-γ delays and out-of-channel effects.

β-γ coincidence counting



β-γ coincidence counting



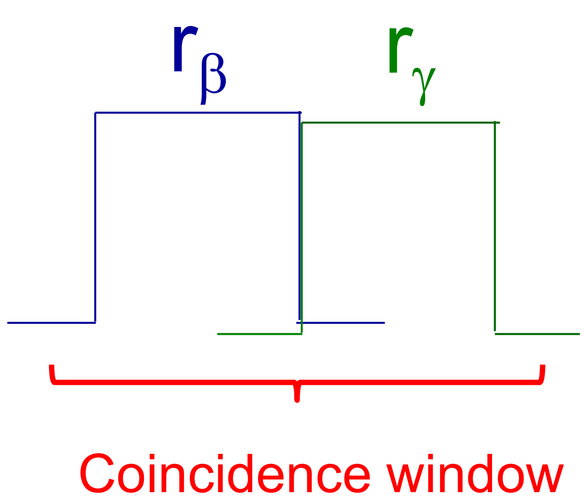
γ after β

β after γ

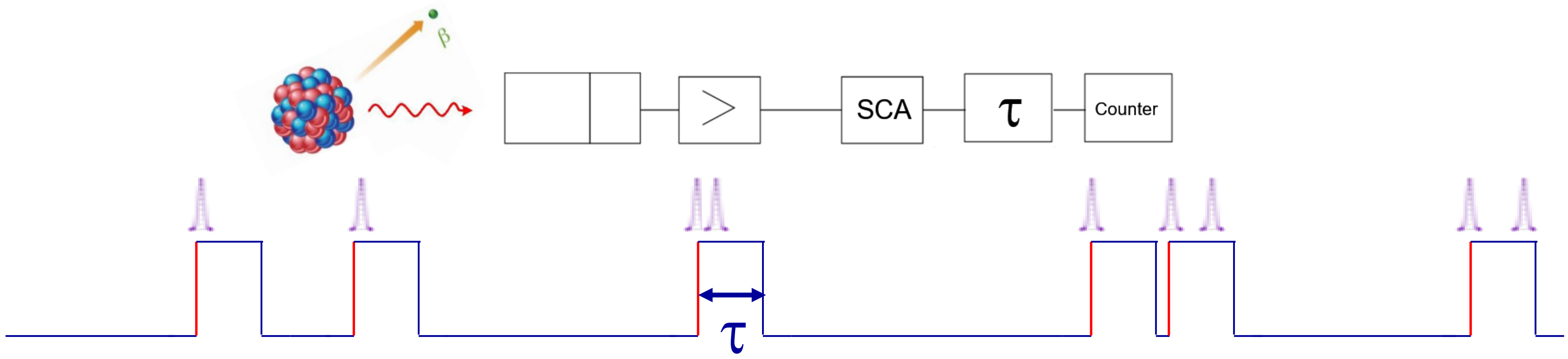
β-γ coincidence counting

- To count all β and γ pulses in genuine coincidence, β- and γ-pulse widths are chosen to be large enough for them to overlap if they are partners in a genuine coincidence.

  If a β-pulse arrives before a coincident γ, make sure its width overlaps with the incoming γ.



Non-extendable deadtime



R : observed countrate

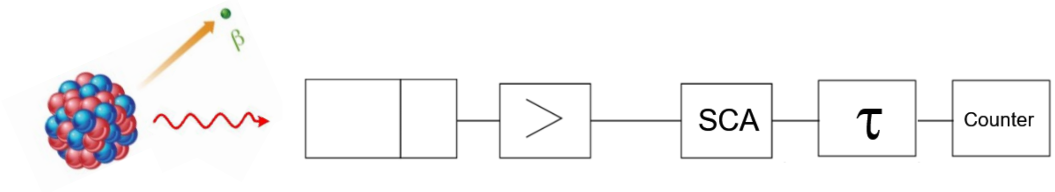
ρ : true countrate

$$\rho = \frac{R}{1-\tau R} \qquad R = \frac{\varrho}{1+\tau\varrho}$$

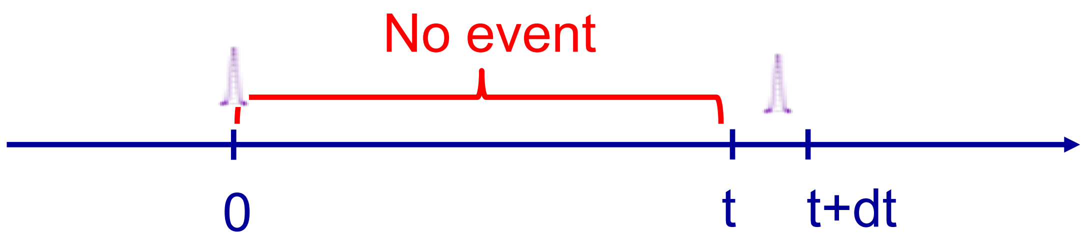
$$p = \frac{R}{\varrho} = \frac{1}{1+\tau\varrho} = 1 - \tau R$$ Prob. of counter being open

$$1 - p = \tau R$$ Prob. of counter being blocked

Poisson process



ρ : true countrate



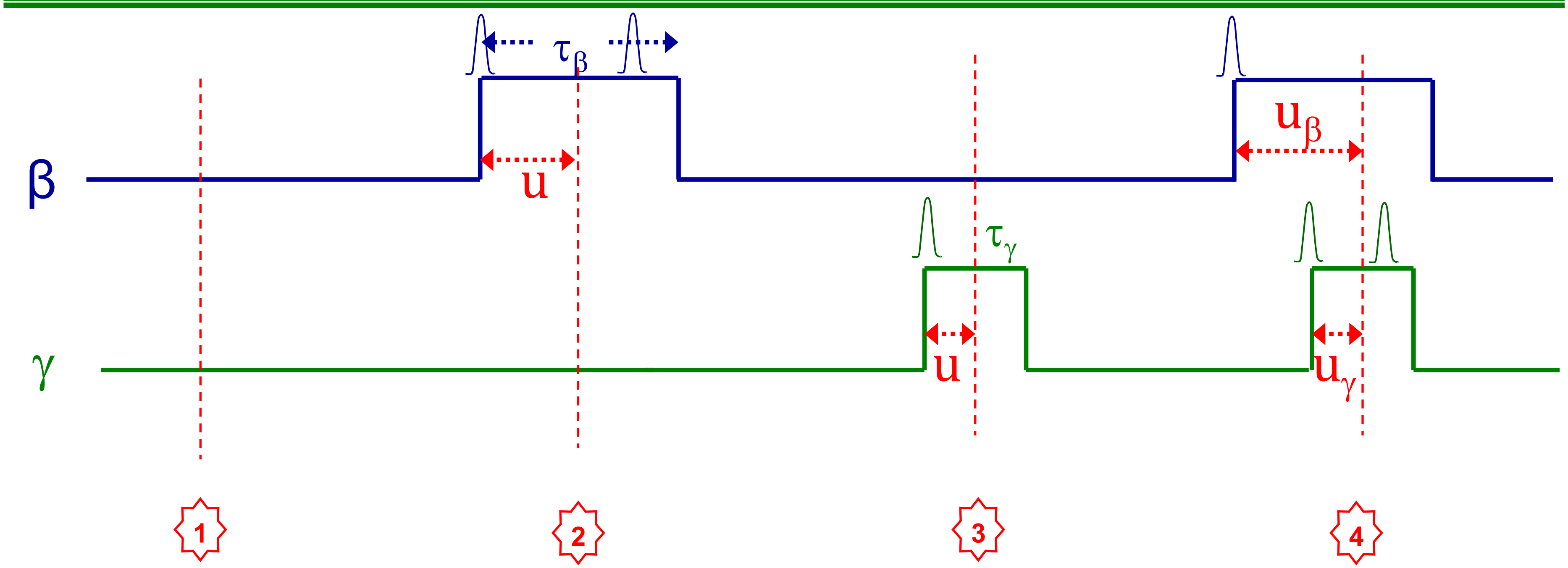
What is the probability that the next event will occur within dt after t ?

$$p = e^{-\rho t} \cdot \rho dt$$

Probability of no pulse during t ← $e^{-\rho t}$

ρdt → Probability of a pulse during dt

CIS coincidence correction



β-counter open
γ-counter open

β-counter locked

γ-counter locked

β-counter shut
γ-counter shut

Probability
$p_{\beta\gamma}$

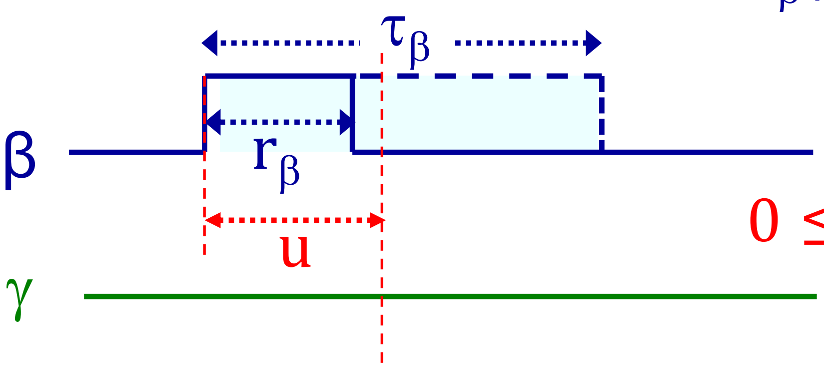
Probability density
$q_{\beta}(u)$
$u \in [0, \tau_{\beta}]$

Prob. dens.
$q_{\gamma}(u)$
$u \in [0, \tau_{\gamma}]$

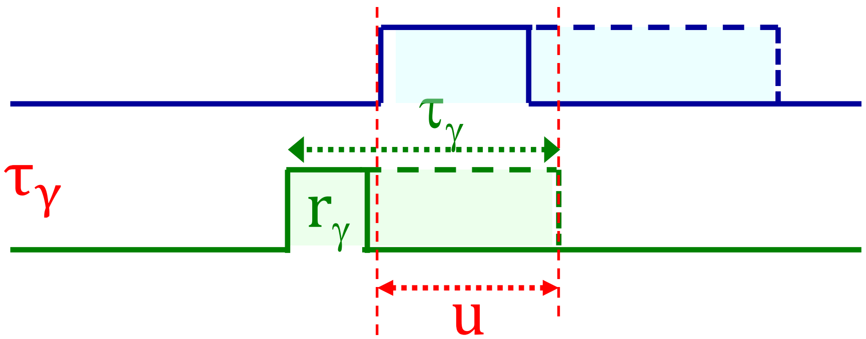
Prob. dens.
$q_{\beta\gamma}(u_{\beta}, u_{\gamma})$
$u_{\beta} \in [0, \tau_{\beta}]$
$u_{\gamma} \in [0, \tau_{\gamma}]$

CIS coincidence correction

- Time evolution of $q_\beta(u)$?



$$0 \le u \le \tau_\gamma$$



γ-counter open

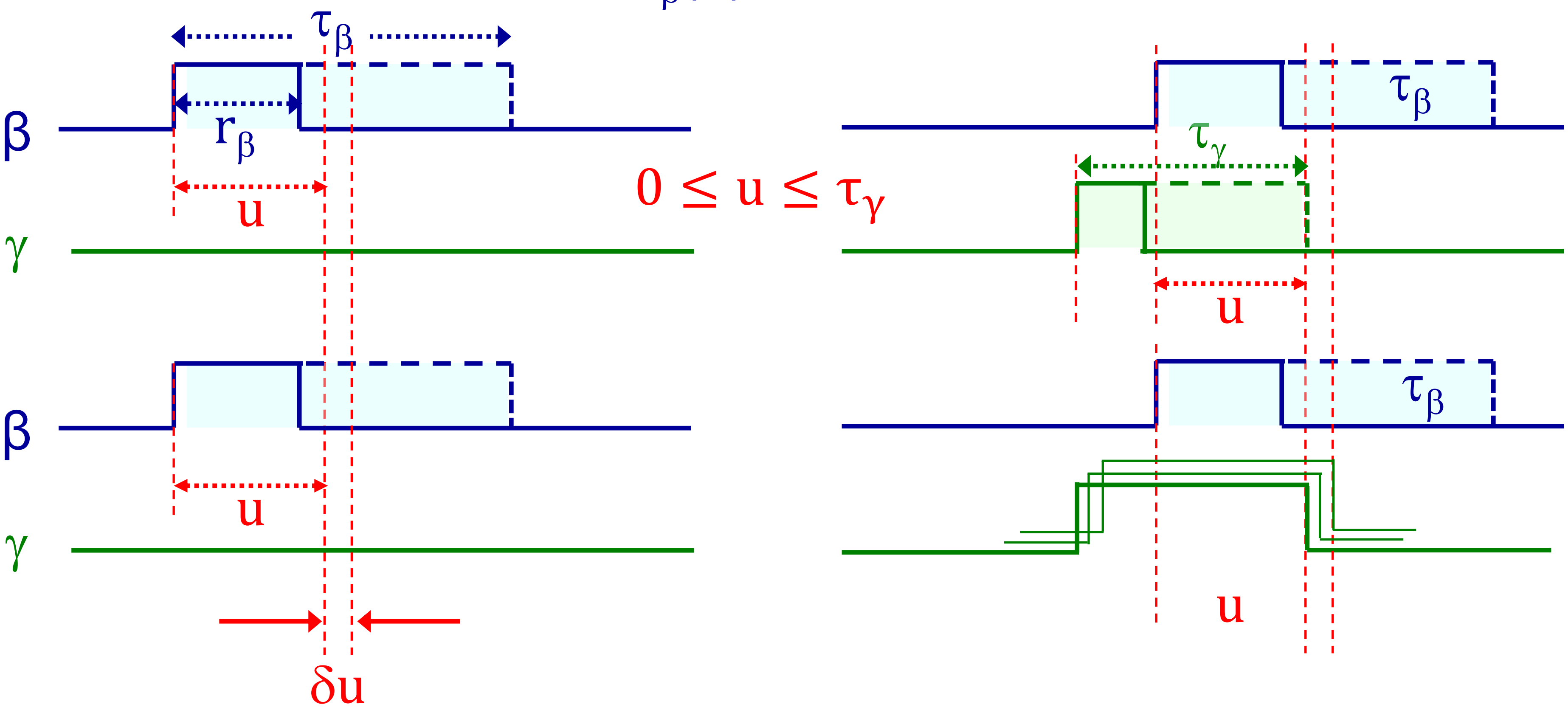
β-counter has been locked for a duration u before the instant of interest

γ-counter becomes open after completing τ_γ just before instant of interest

β-counter has been locked for duration u.

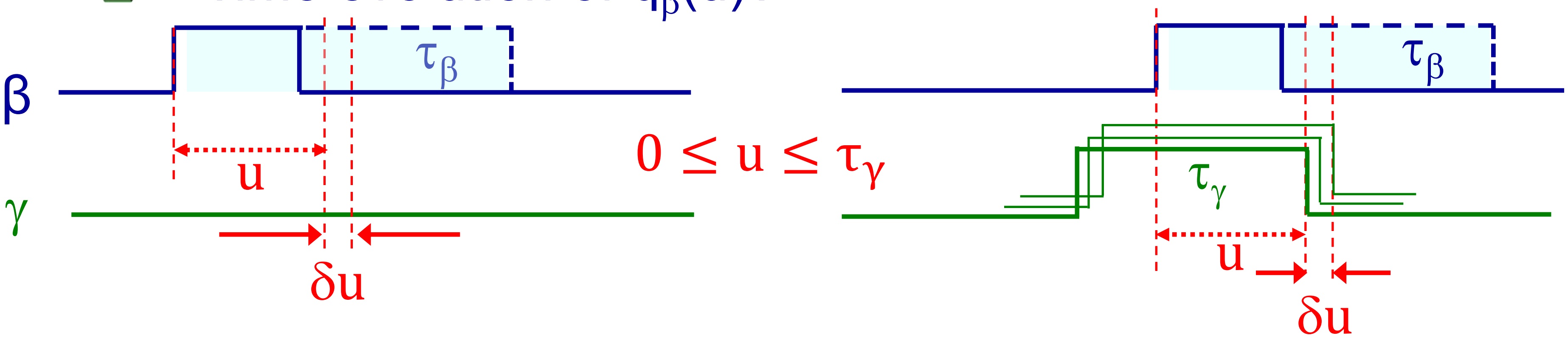
CIS coincidence correction

- Time evolution of $q_\beta(u)$?



CIS coincidence correction

- Time evolution of $q_\beta(u)$?



$q_\beta(u+\delta u)$ can be expressed in terms of $q_\beta(u)$ and changes or otherwise in the state of the system during [u, u+δu].

$$q_\beta(u+\delta u) = q_\beta(u)\cdot e^{-\rho_\gamma \delta u} + q_{\beta\gamma}(u,\tau_\gamma)\delta u$$

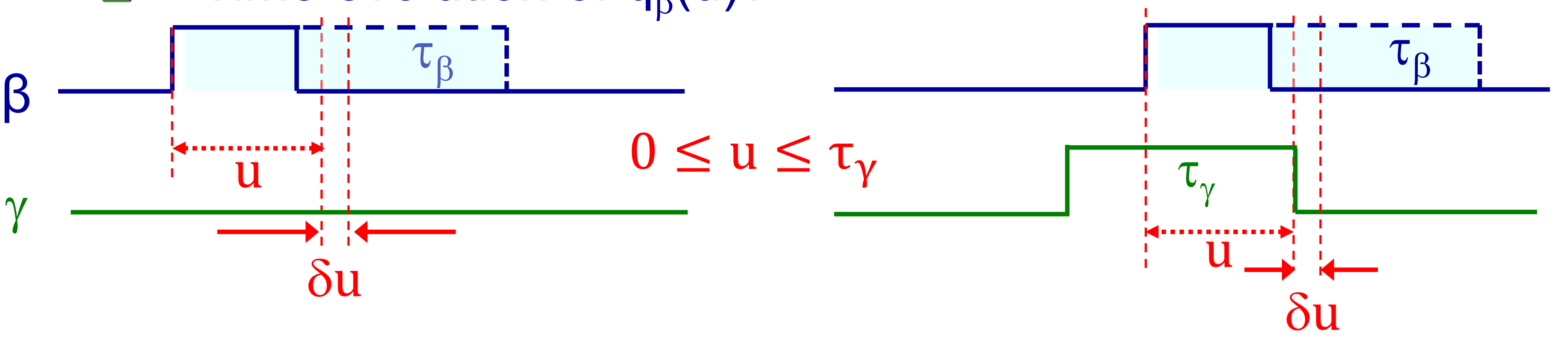
Probability of no pulse in γ-channel during δu

Prob. density that 2-counters are locked,

β : u, and γ: τ_γ.

CIS coincidence correction

- Time evolution of $q_\beta(u)$?



$q_\beta(u + \delta u)$ can be expressed in terms of $q_\beta(u)$ and changes or otherwise in the state of the system during $[u, u+\delta u]$.

$$q_\beta(u + \delta u) = q_\beta(u) \cdot e^{-\rho_\gamma \delta u} + q_{\beta\gamma}(u, \tau_\gamma)\delta u$$

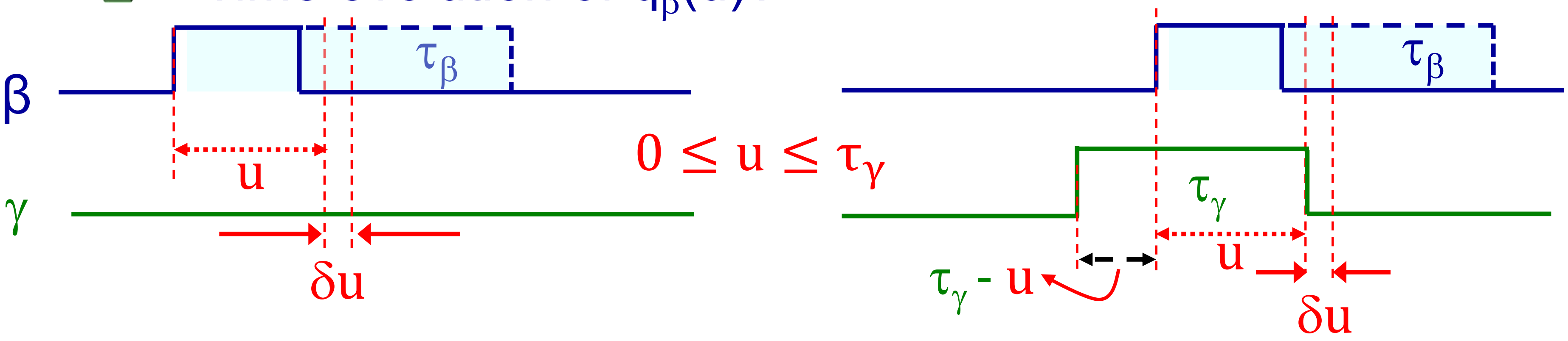
Probability of no pulse in γ-channel during δu

Prob. density that 2-counters are locked,

β : u, and γ: τ_γ.

CIS coincidence correction

- Time evolution of $q_{\beta}(u)$?



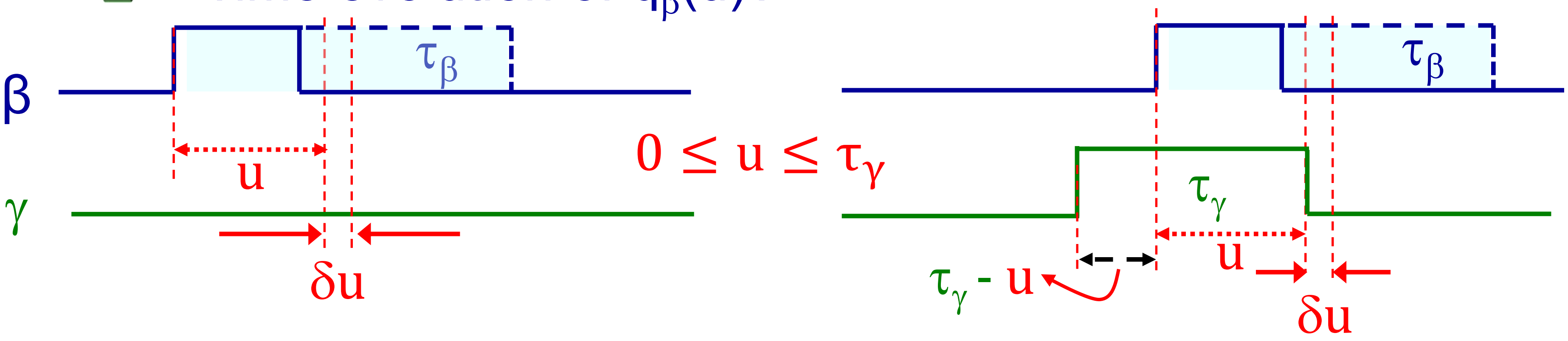
$$q_{\beta\gamma}(u, \tau_\gamma) = q_\gamma(\tau_\gamma - u)\, \delta u\, \rho_\beta$$

Prob. dens. that β-counter is open while γ-counter has been locked for τ_γ - u

Prob. of β-pulse during δu

CIS coincidence correction

- Time evolution of $q_\beta(u)$?



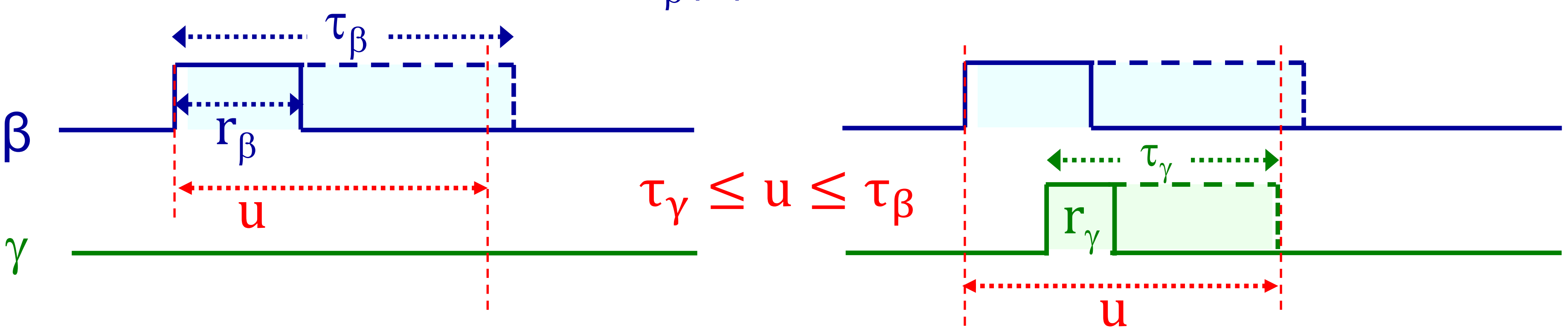
$$q_\beta(u+\delta u) = q_\beta(u)\cdot e^{-\rho_\gamma \delta u} + q_\gamma(\tau_\gamma - u)\,\delta u\,\rho_\beta$$

$$= q_\beta\,(u)\left(1-\rho_\gamma \delta u\right) + q_\gamma(\tau_\gamma - u)\,\delta u\,\rho_\beta$$

$$\lim_{\delta u \to 0} \frac{q_\beta(u+\delta u) - q_\beta(u)}{\delta u} = q'_\beta(u) = -q_\beta(u)\rho_\gamma + q_\gamma(\tau_\gamma - u)\rho_\beta$$

CIS coincidence correction

- Time evolution of $q_\beta(u)$?



γ-counter open

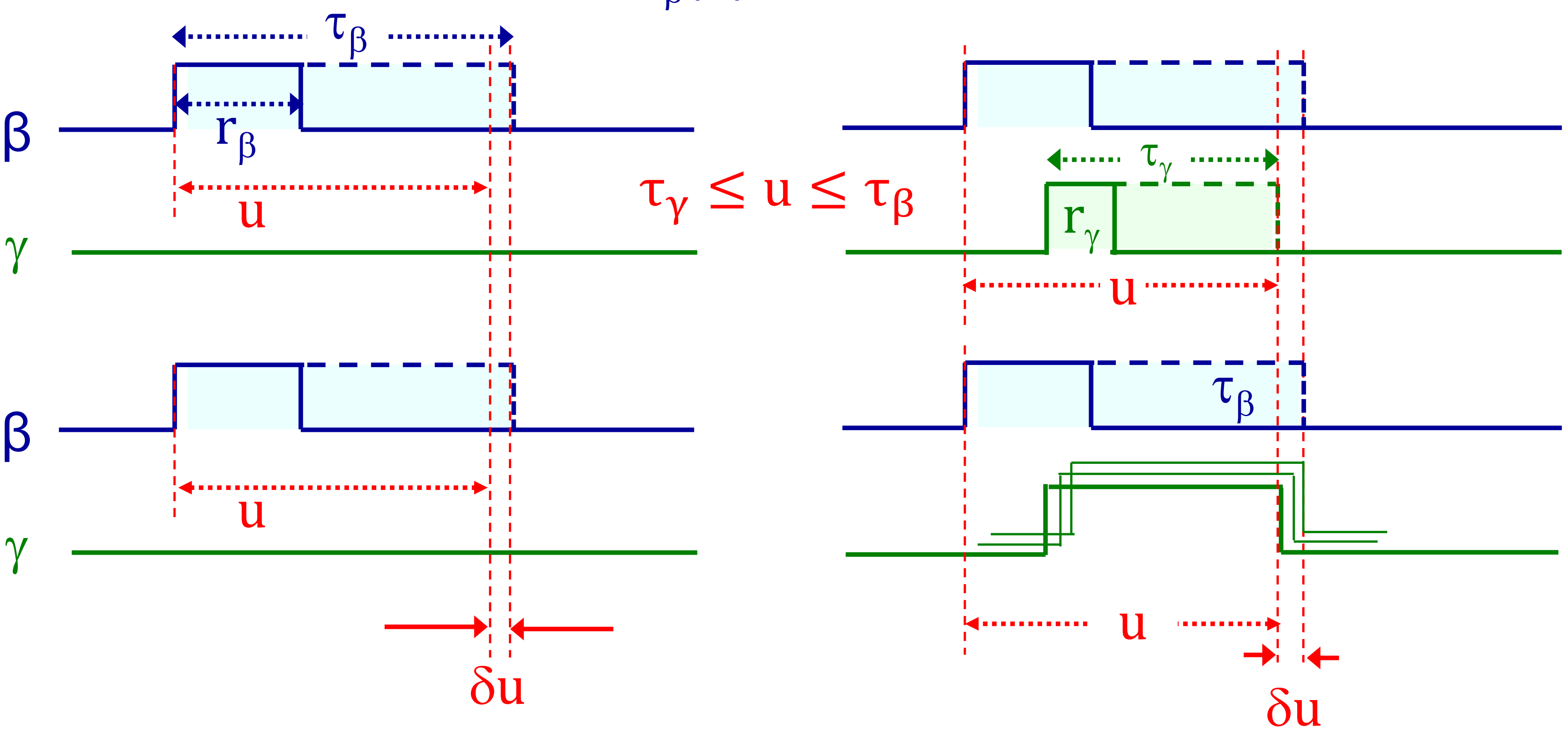
β-counter has been locked for a duration u before the instant of interest

γ-counter becomes open after completing τ_γ just before instant of interest

β-counter has been locked for duration u.

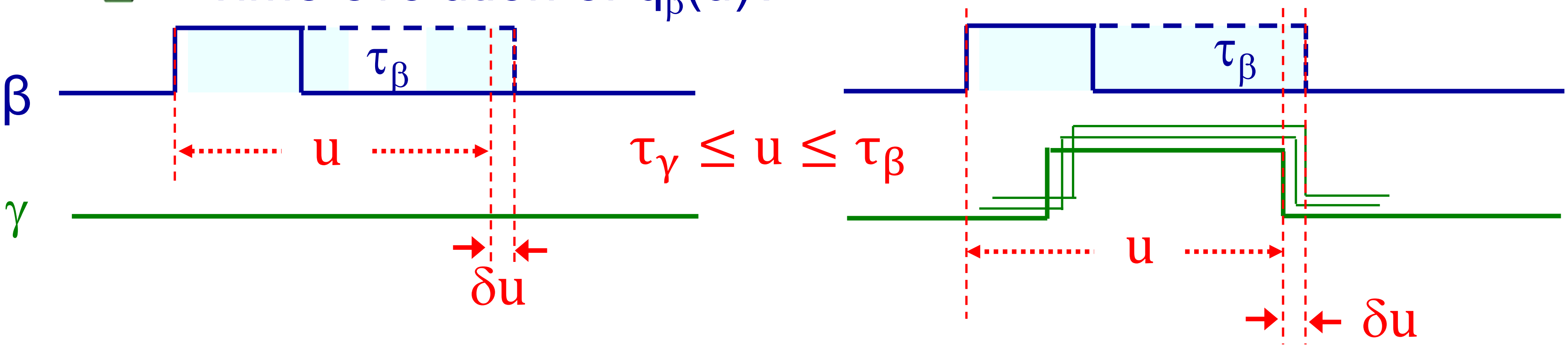
CIS coincidence correction

- Time evolution of $q_\beta(u)$?



CIS coincidence correction

- Time evolution of $q_\beta(u)$?



$$q_\beta(u+\delta u) = q_\beta(u)\cdot e^{-\rho_\gamma \delta u} + q_{\beta\gamma}(u,\tau_\gamma)\delta u$$

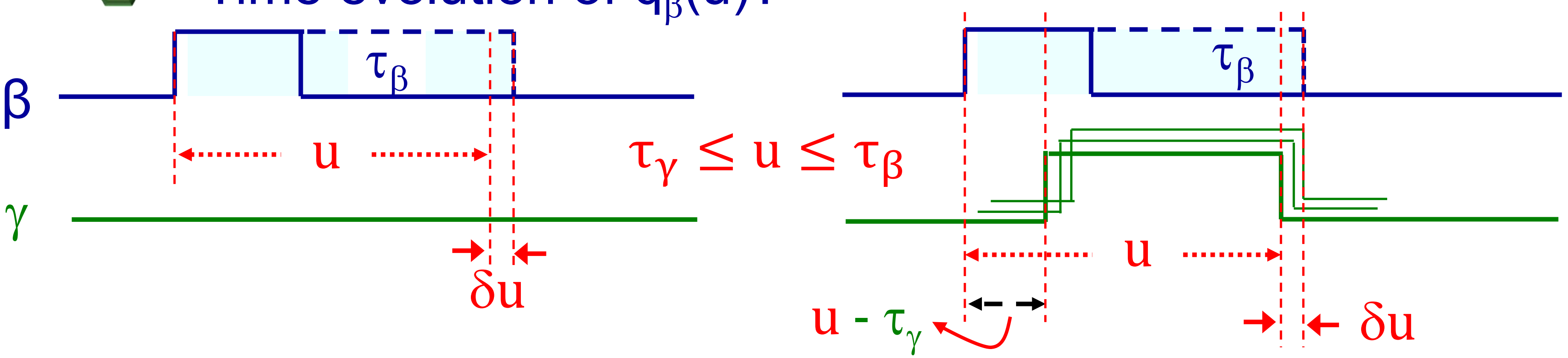
Probability of no pulse in γ-channel during δu

Prob. density that 2-counters are locked,

β : u, and γ: τ_γ.

CIS coincidence correction

- Time evolution of $q_\beta(u)$?



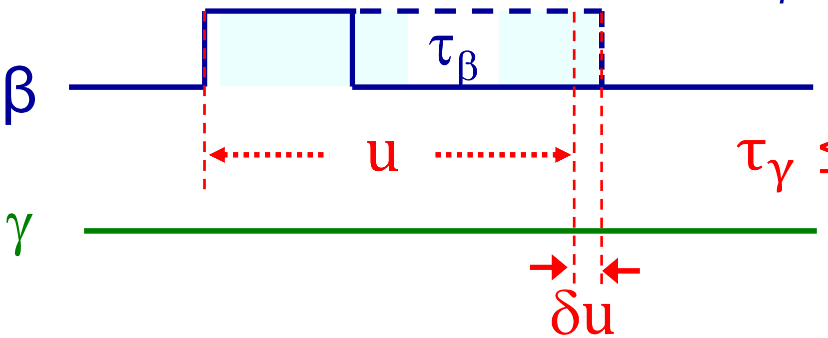
$$q_{\beta\gamma}(u,\tau_\gamma) = q_\beta(u-\tau_\gamma)\,\delta u\,\rho_\gamma$$

Prob. dens. that γ-counter is open while β-counter has been locked for u - τ_γ, within δu

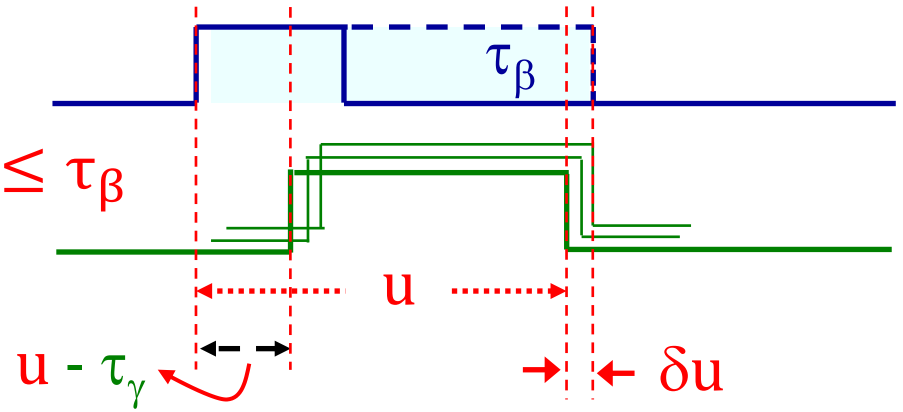
Prob. of γ-pulse during δu

CIS coincidence correction

- Time evolution of $q_\beta(u)$?



$\tau_\gamma \le u \le \tau_\beta$



$$q_\beta(u+\delta u) = q_\beta(u)\cdot e^{-\rho_\gamma \delta u} + q_\beta(u-\tau_\gamma)\,\delta u\,\rho_\gamma$$

$$= q_\beta\,(u)(1-\rho_\gamma\delta u) + q_\beta(u-\tau_\gamma)\,\delta u\,\rho_\gamma$$

$$q'_\beta(u) = -q_\beta(u)\rho_\gamma + q_\beta(u-\tau_\gamma)\rho_\gamma$$

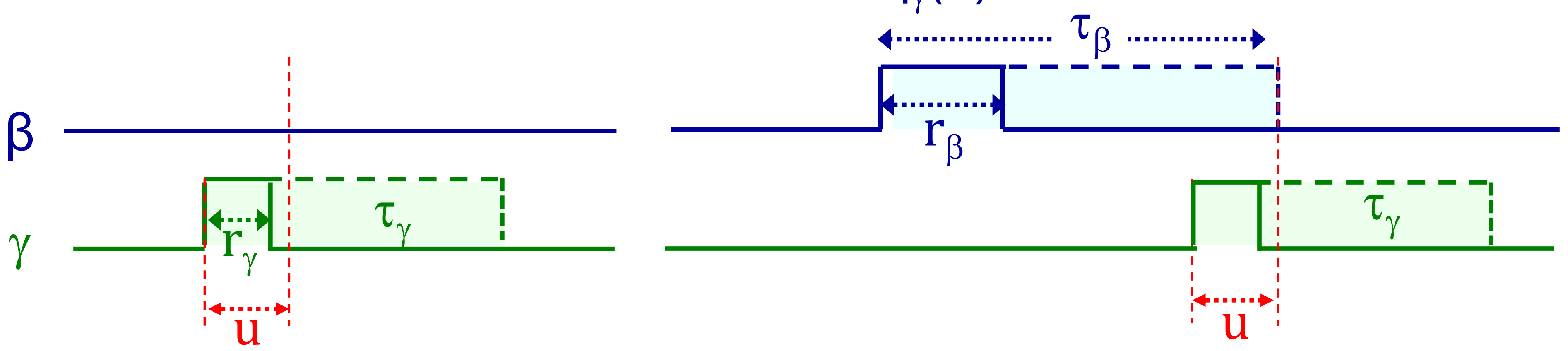
CIS coincidence correction

- Time evolution of $q_\beta(u)$?

$$q'_\beta(u) = -q_\beta(u)\rho_\gamma + \begin{cases} q_\gamma(\tau_\gamma - u)\varrho_\beta & \text{if} \quad 0 \le u \le \tau_\gamma \\ q_\beta(u - \tau_\gamma)\varrho_\gamma & \text{if} \quad \tau_\gamma \le u \le \tau_\beta \end{cases}$$

CIS coincidence correction

- What is the time evolution of $q_\gamma(u)$?



β-counter open

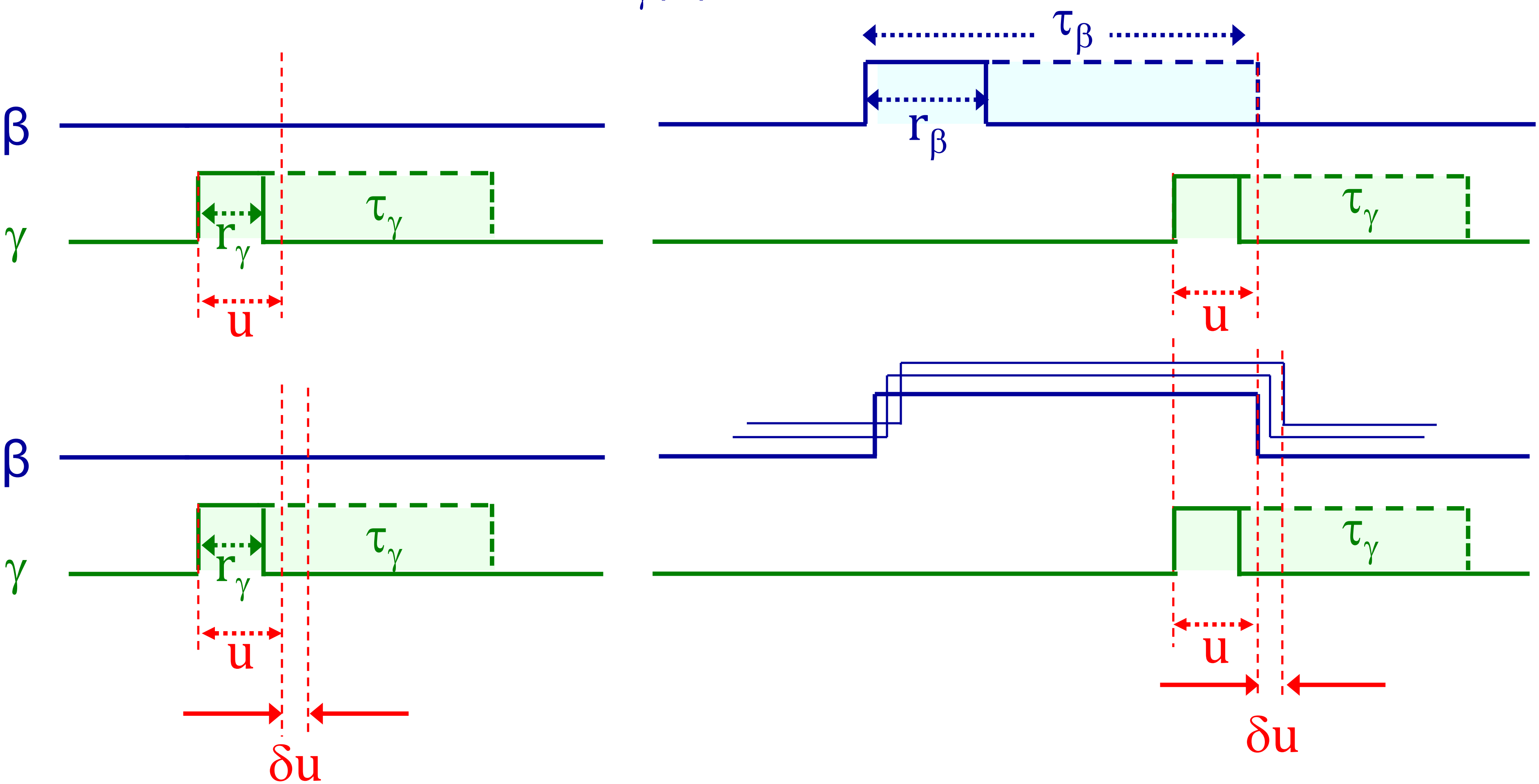
γ-counter has been locked for a duration u before instant of instant of interest

β-counter becomes open after completing τ_β just before instant of interest

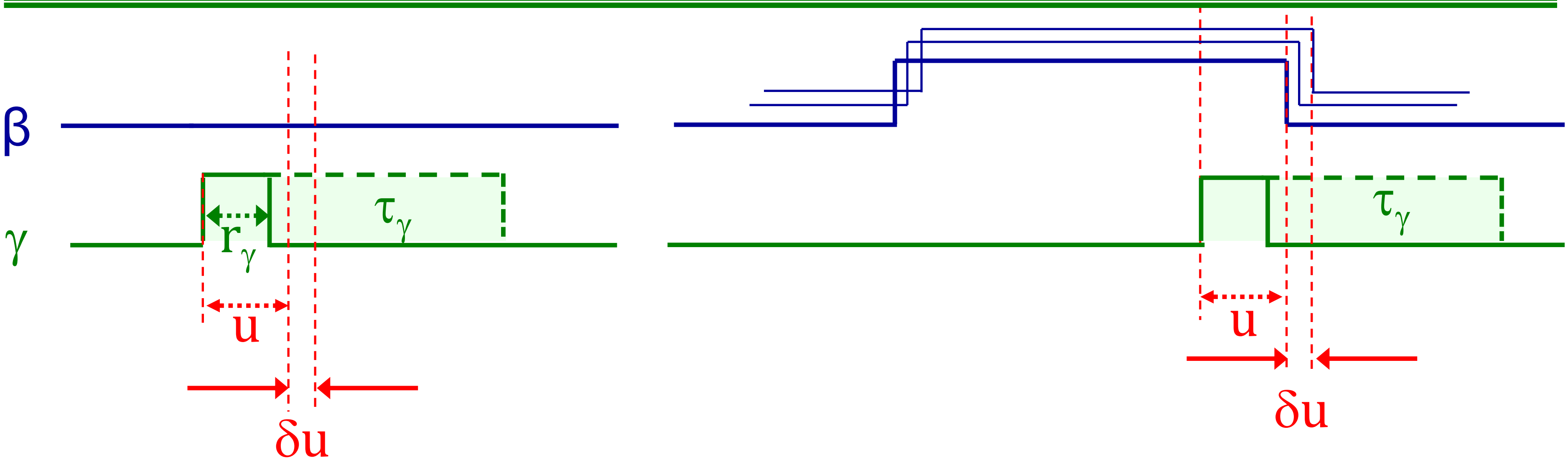
γ-counter has been locked for duration u.

CIS coincidence correction

- Time evolution of $q_\gamma(u)$?



CIS coincidence correction



$q_\gamma(u+\delta u)$ can be expressed in terms of $q_\gamma(u)$ and changes or otherwise in the state of the system during [u, u+δu].

$$q_\gamma(u+\delta u) = q_\gamma(u)\cdot e^{-\rho_\beta \delta u} + q_{\beta\gamma}(\tau_\beta, u)\delta u$$

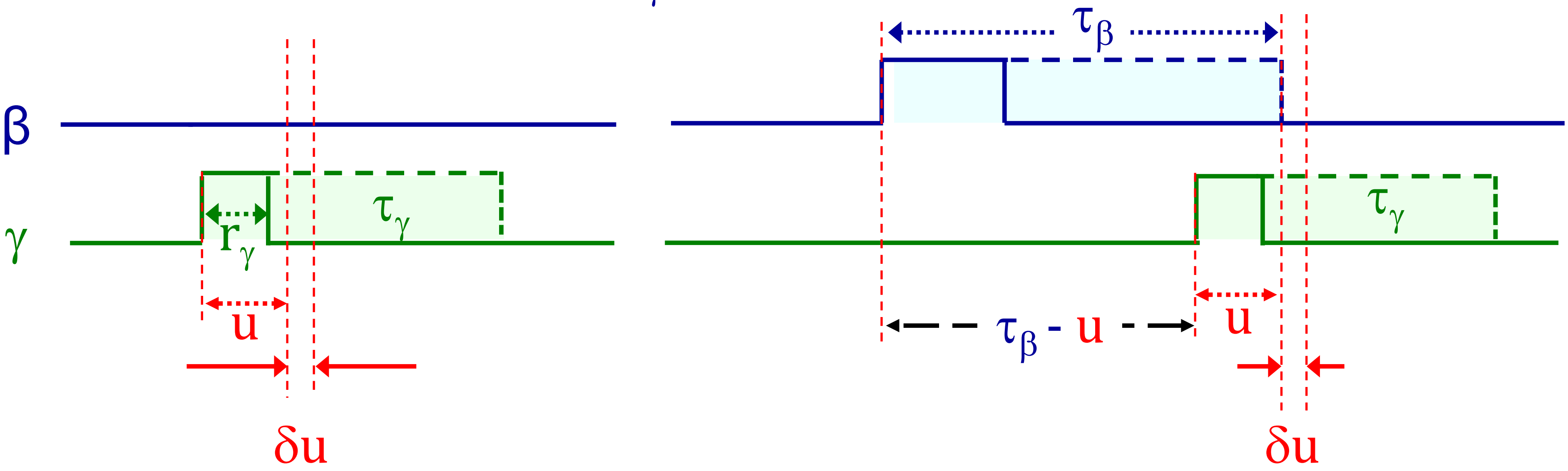
Probability of no pulse in β-channel during δu

Prob. density that 2-counters are locked,

β : τ_β, and γ: u.

CIS coincidence correction

- Time evolution of $q_\gamma(u)$?

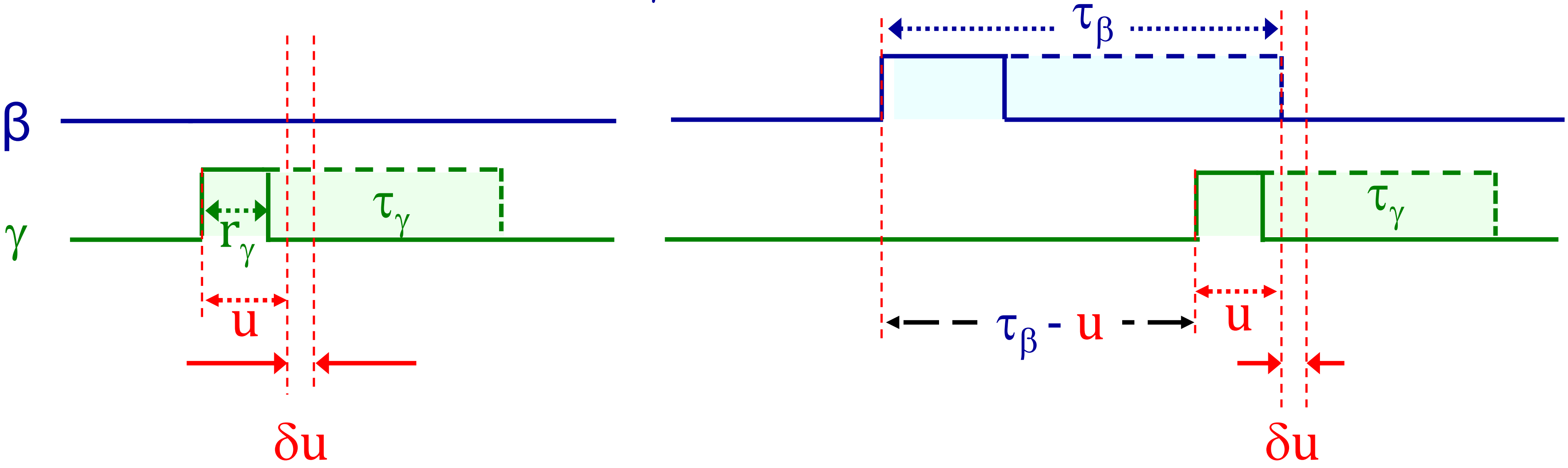


$$q_{\beta\gamma}(\tau_\beta, u) = q_\beta(\tau_\beta - u)\,\delta u\,\rho_\gamma$$

Prob. that γ-counter is open while β-counter has been locked for τ_β - u, within δu

CIS coincidence correction

- Time evolution of $q_\gamma(u)$?



$$q'_\gamma(u) = -q_\gamma(u)\rho_\beta + q_\beta(\tau_\beta - u)\rho_\gamma$$

CIS coincidence correction

- Summary

$$\begin{cases} q'_{\beta}(u) = -q_{\beta}(u)\rho_{\gamma} + \begin{cases} q_{\gamma}(\tau_{\gamma} - u)\varrho_{\beta} & \text{if} \quad 0 \leq u \leq \tau_{\gamma} \\ q_{\beta}(u - \tau_{\gamma})\varrho_{\gamma} & \text{if} \quad \tau_{\gamma} \leq u \leq \tau_{\beta} \end{cases} \\ \\ q'_{\gamma}(u) = -q_{\gamma}(u)\rho_{\beta} + q_{\beta}(\tau_{\beta} - u)\rho_{\gamma} \end{cases}$$

Coupled differential equations are constrained by boundary, continuity and normalisations conditions.

CIS coincidence correction

- Boundary condition

$$q_\beta(0) = p_{\beta\gamma}\left(\varrho_\beta - \varrho_c\right) \qquad q_\gamma(0) = p_{\beta\gamma}\left(\varrho_\gamma - \varrho_c\right)$$

Prob. dens. that β-counter just blocks while γ-counter is open.

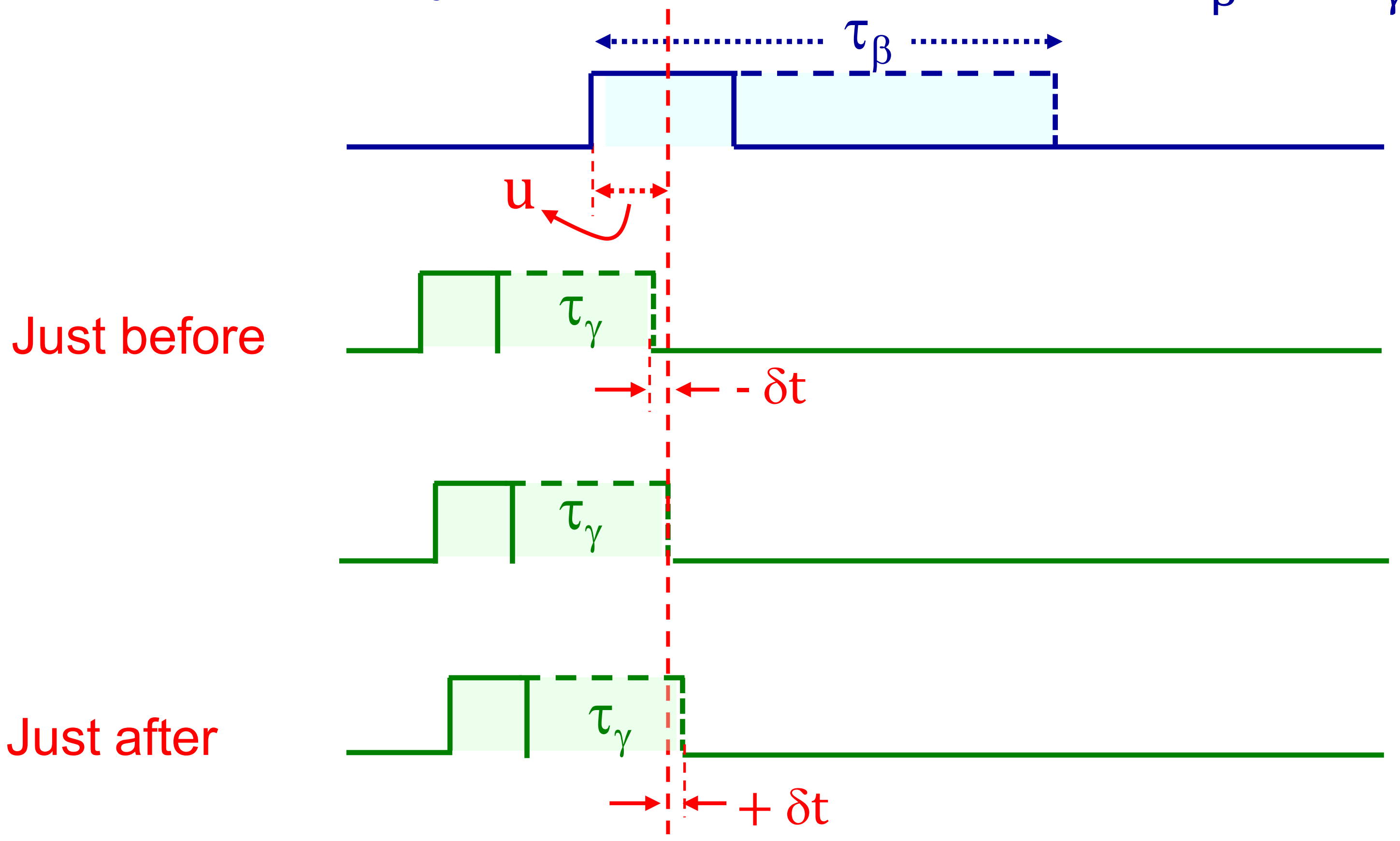
Prob. that both counters are open

Rate of unpaired β-pulses

CIS coincidence correction

- Continuity condition

$\tau_\beta = 2\tau_\gamma$

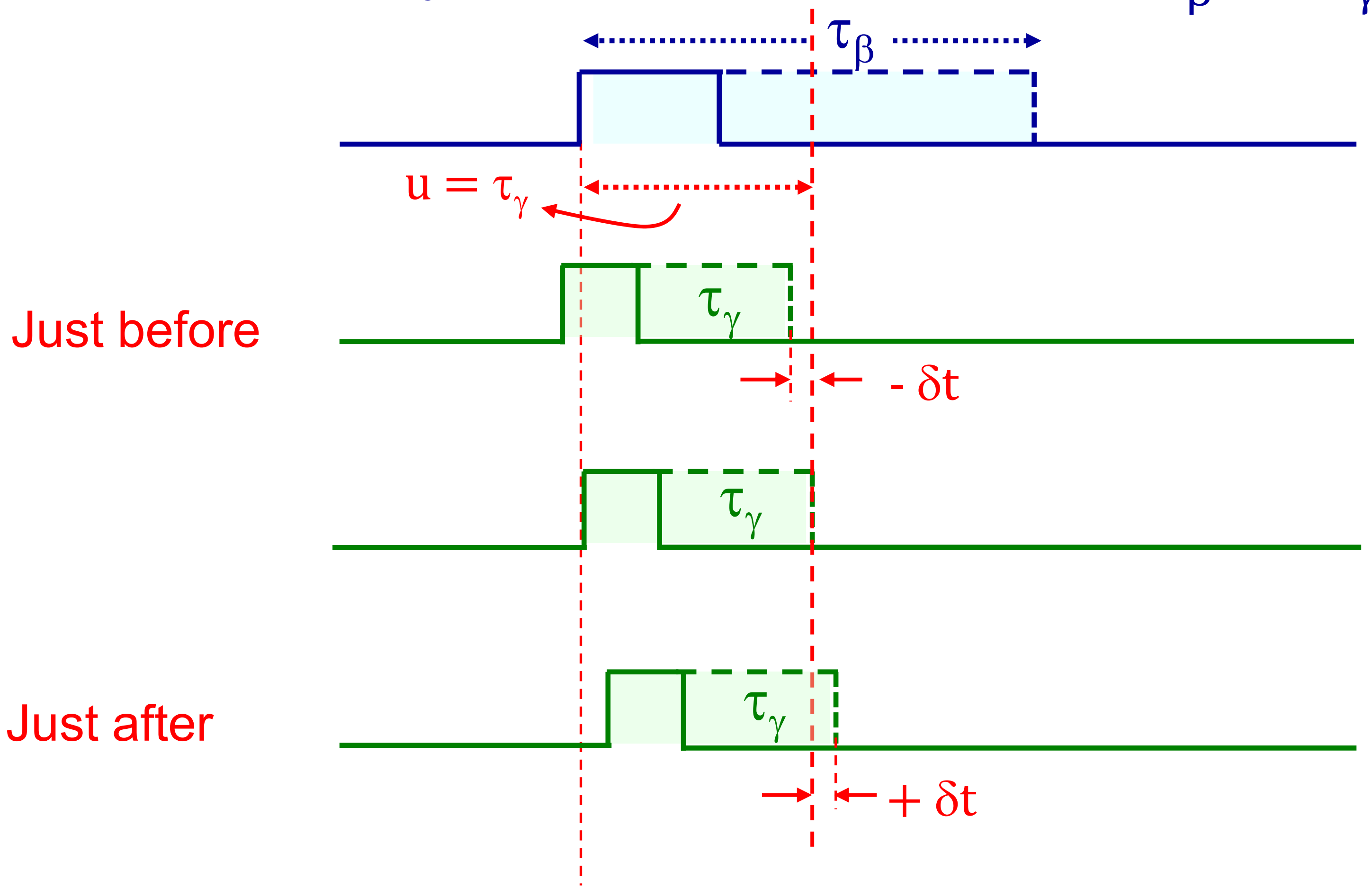


$$q_\beta(u + \delta t) = q_\beta(u - \delta t)$$

CIS coincidence correction

- Continuity condition

$$\tau_\beta = 2\tau_\gamma$$



$$q_\beta(\tau_\gamma + \delta t) = q_\beta(\tau_\gamma - \delta t) + p_{\beta\gamma}\varrho_c$$

CIS coincidence correction

- Normalisation condition

$$p_\beta = p_{\beta\gamma} + \int_0^{\tau_\gamma} q_\gamma(u)\,du$$

Probability the β-counter is open (p_β)

Prob. dens. γ-counter is locked when β-counter is open. ($q_\gamma(u)$)

$$p_\gamma = p_{\beta\gamma} + \int_0^{\tau_\beta} q_\beta(u)\,du$$

CIS coincidence correction

- Observed coincidence rate

$$R_c = p_{\beta\gamma}\varrho_c + R_f$$

$$R_f = p_{\beta\gamma}(\varrho_\beta - \varrho_c)\int_0^{r_\beta} e^{-\varrho_\gamma x}\varrho_\gamma dx \; + \; p_{\beta\gamma}(\varrho_\gamma - \varrho_c)\int_0^{r_\gamma} e^{-\varrho_\beta x}\varrho_\beta dx \; +$$

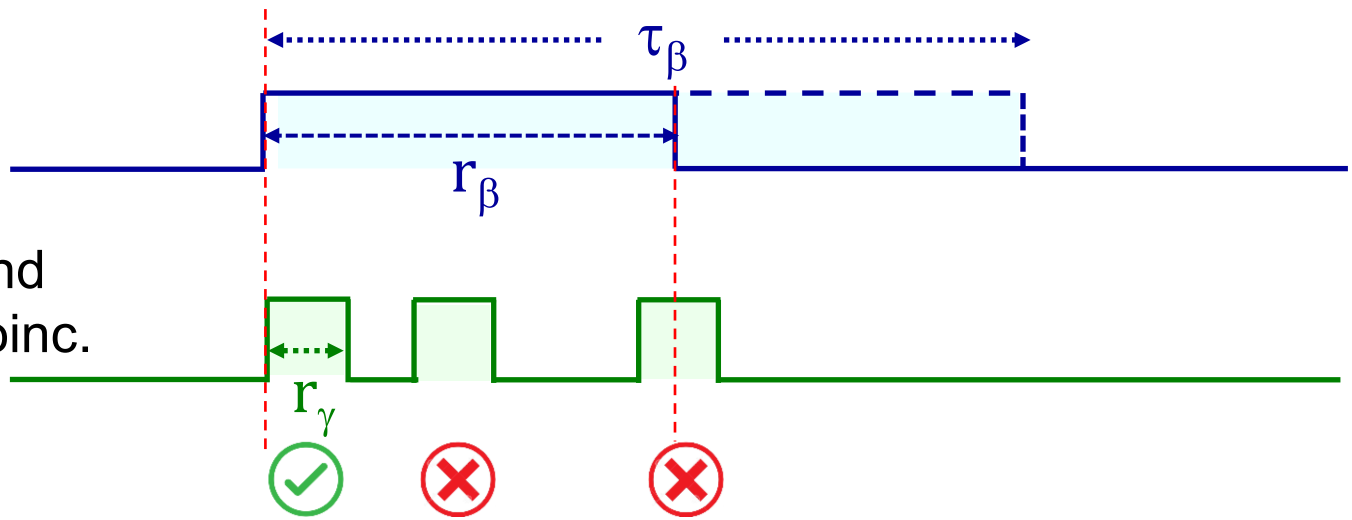
$$\int_{\tau_\gamma - r_\beta}^{\tau_\gamma} q_\gamma(u)\varrho_\beta\left[\int_0^{u + r_\beta - \tau_\gamma} e^{-\varrho_\gamma y}\varrho_\gamma dy\right] du \; +$$

$$\int_{\tau_\beta - r_\gamma}^{\tau_\beta} q_\beta(u)\varrho_\gamma\left[\int_0^{u + r_\gamma - \tau_\beta} e^{-\varrho_\beta y}\varrho_\beta dy\right] du$$

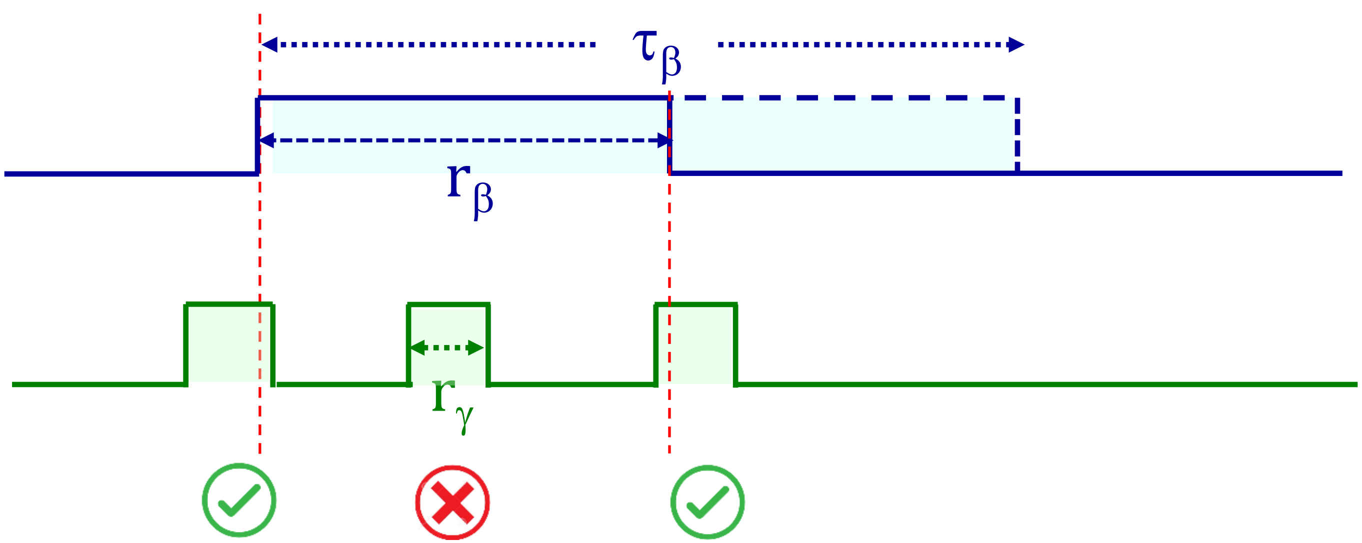
CIS coincidence correction

- Assumptions about fortuitous coincidences

A genuine coincidence, and no fortuitous coinc.



Two fortuitous coinc. at most



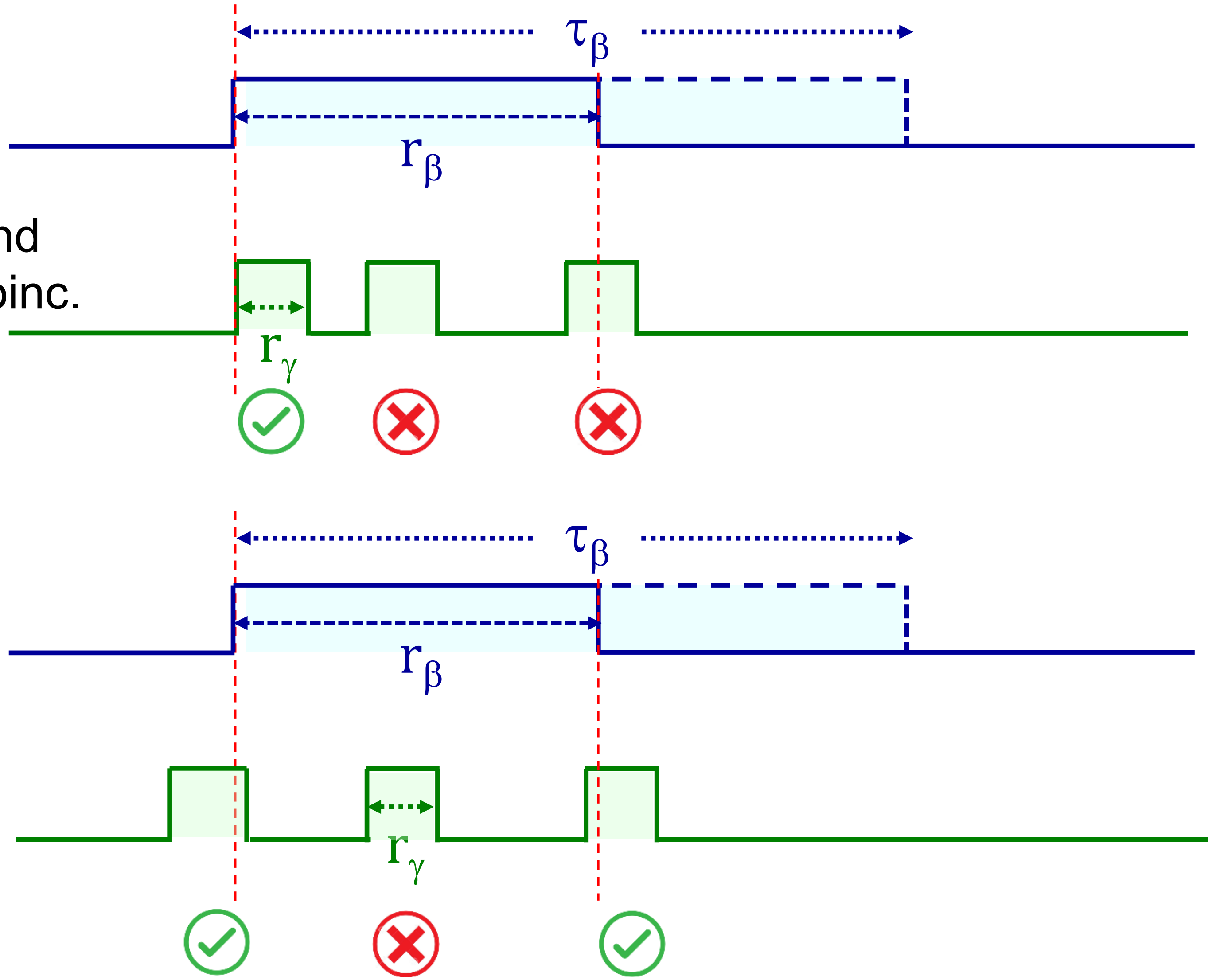
CIS coincidence correction

- Assumptions

A genuine coincidence, and no fortuitous coinc.

$\tau_\gamma \geq r_\beta$

Two fortuitous coinc. at most



CIS coincidence correction

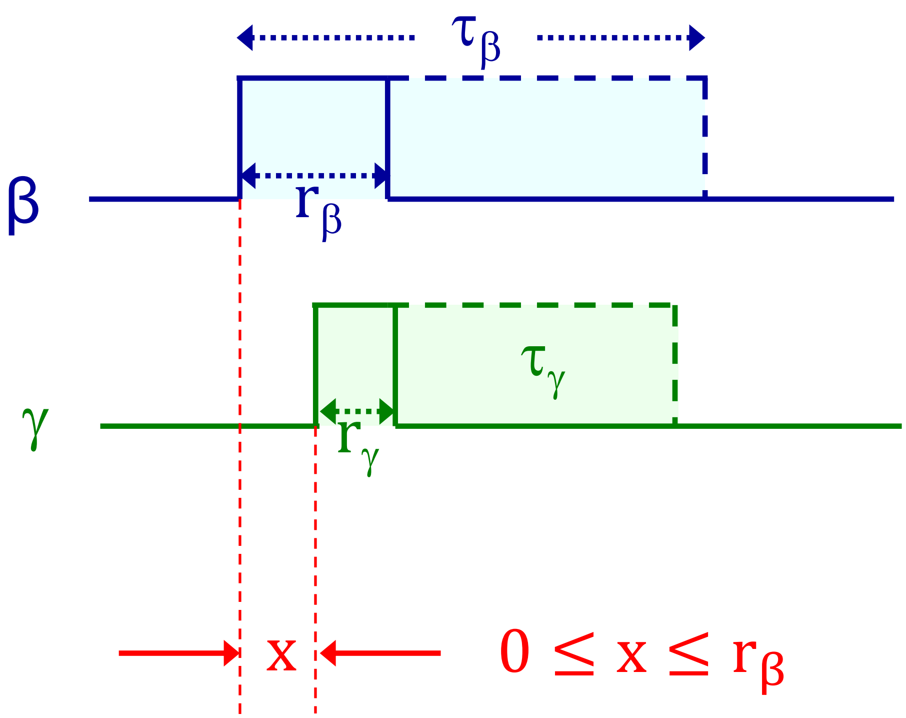
- Assumptions
  - $\tau_\gamma \geq r_\beta$ and $\tau_\beta \geq r_\gamma$
  - $\tau_\gamma \geq r_\gamma$ and $\tau_\beta \geq r_\beta$

$$\max(r_\beta, r_\gamma) \leq \min\left(\tau_\beta, \tau_\gamma\right)$$

$$r_\beta + r_\gamma \leq \min\left(\tau_\beta, \tau_\gamma\right)$$

CIS coincidence correction

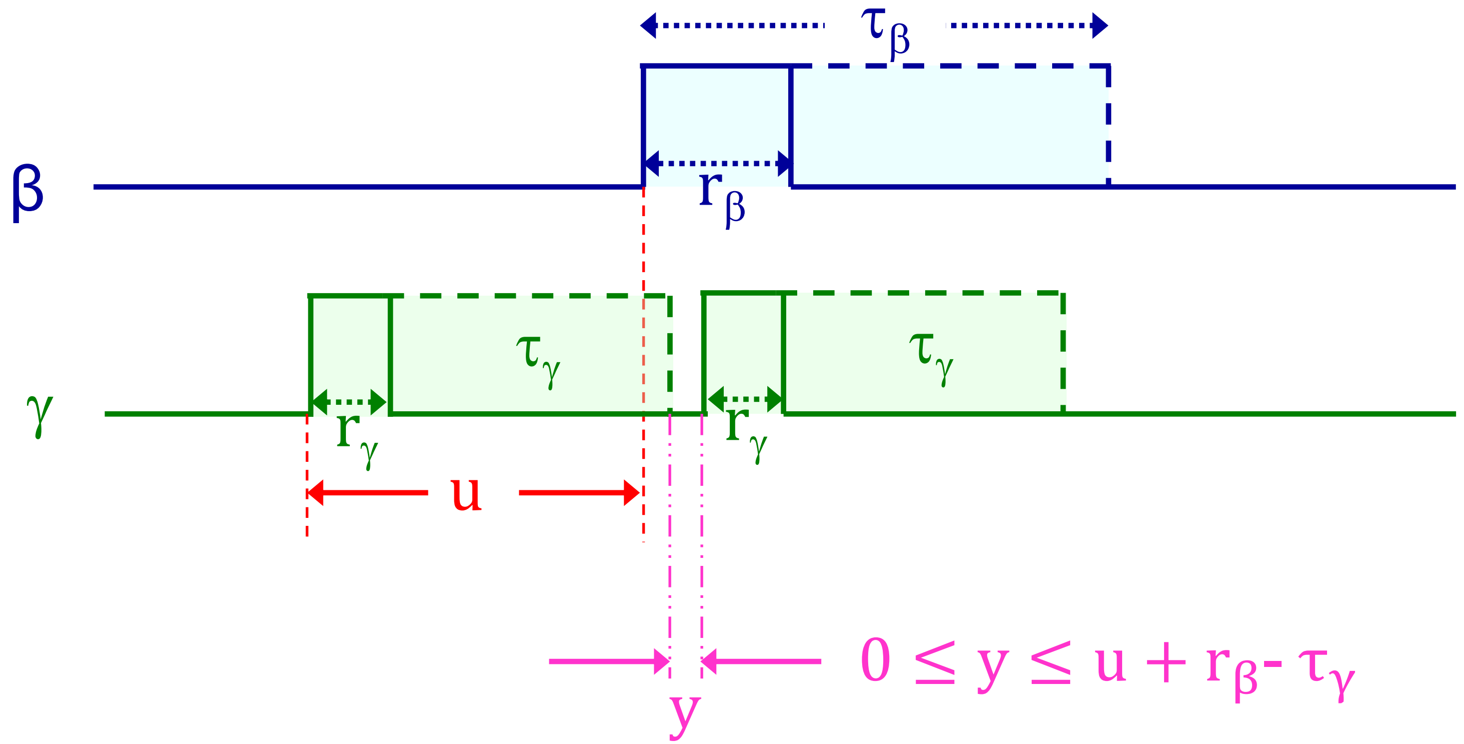
- Type 1 fortuitous coincidences



$$\left(\varrho_\beta - \varrho_c\right)\int_0^{r_\beta} e^{-\varrho_\gamma x}\varrho_\gamma dx$$

CIS coincidence correction

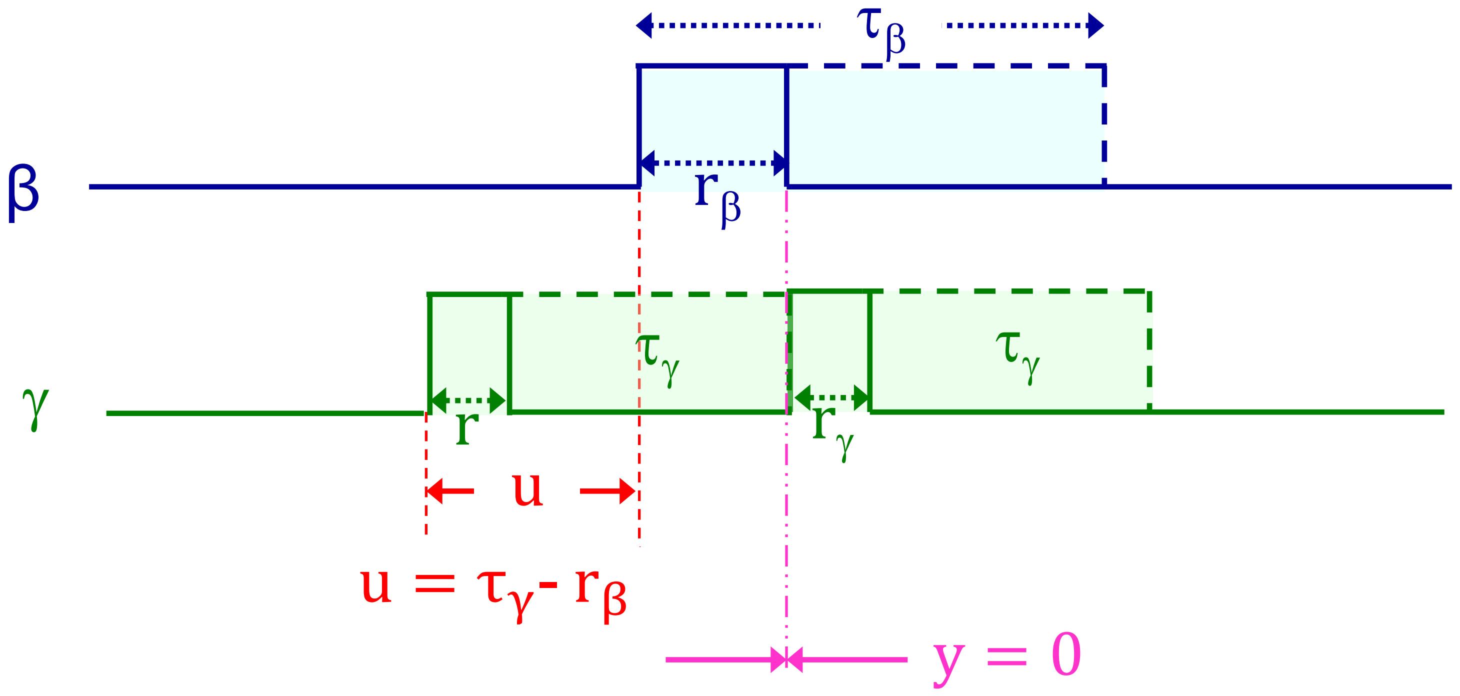
- Type 2 fortuitous coincidences



$$\int_{\tau_\gamma - r_\beta}^{\tau_\gamma} q_\gamma(u)\varrho_\beta \left[\int_0^{u+r_\beta-\tau_\gamma} e^{-\varrho_\gamma y}\varrho_\gamma dy\right] du$$

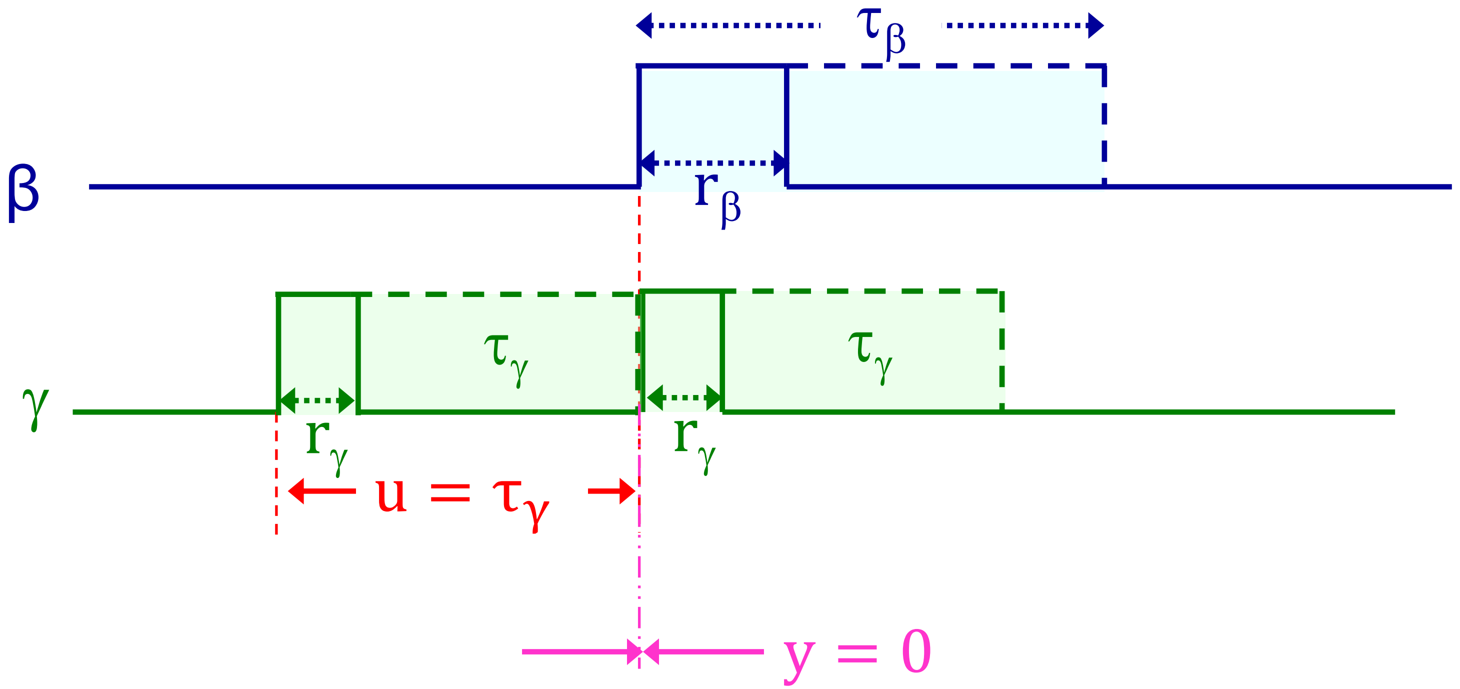
CIS coincidence correction

- Type 2 fortuitous coincidences



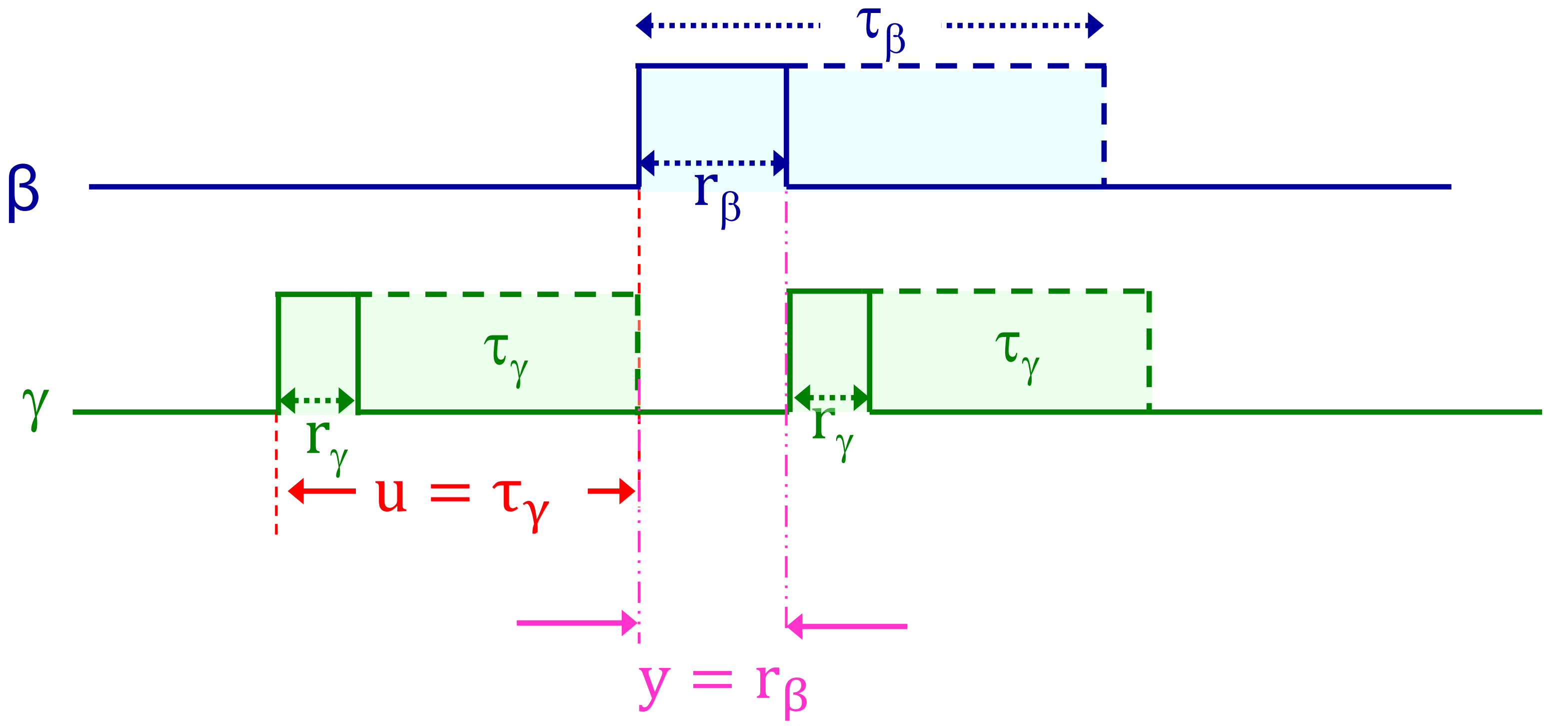
CIS coincidence correction

- Type 2 fortuitous coincidences



CIS coincidence correction

- Type 2 fortuitous coincidences



CIS coincidence correction

- Observed coincidence rate

$$R_c = p_{\beta\gamma}\varrho_c + p_{\beta\gamma}\left(\varrho_\beta - \varrho_c\right)\int_0^{r_\beta} e^{-\varrho_\gamma x}\varrho_\gamma dx + \int_{\tau_\gamma - r_\beta}^{\tau_\gamma} q_\gamma(u)\varrho_\beta\left[\int_0^{u+r_\beta-\tau_\gamma} e^{-\varrho_\gamma y}\varrho_\gamma dy\right]du + p_{\beta\gamma}\left(\varrho_\gamma - \varrho_c\right)\int_0^{r_\gamma} e^{-\varrho_\beta x}\varrho_\beta dx + \int_{\tau_\beta - r_\gamma}^{\tau_\beta} q_\beta(u)\varrho_\gamma\left[\int_0^{u+r_\gamma-\tau_\beta} e^{-\varrho_\beta y}\varrho_\beta dy\right]du$$

CIS coincidence correction

- Observed coincidence rate

$$R_c = p_{\beta\gamma}\varrho_c + p_{\beta\gamma}(\varrho_\beta - \varrho_c)(1 - e^{-\varrho_\gamma r_\beta}) +$$

$$\int_{\tau_\gamma - r_\beta}^{\tau_\gamma} q_\gamma(u)\varrho_\beta \left[1 - e^{-\varrho_\gamma(u + r_\beta - \tau_\gamma)}\right] du \; +$$

$$p_{\beta\gamma}(\varrho_\gamma - \varrho_c)(1 - e^{-\varrho_\beta r_\gamma}) +$$

$$\int_{\tau_\beta - r_\gamma}^{\tau_\beta} q_\beta(u)\varrho_\gamma \left[1 - e^{-\varrho_\beta(u + r_\gamma - \tau_\beta)}\right] du$$

CIS coincidence correction

- Particular case $r_\beta = r_\gamma$ and $\tau_\beta = \tau_\gamma$

$$\begin{cases} q'_\beta(u) = -q_\beta(u)\rho_\gamma + q_\gamma(\tau - u)\varrho_\beta \\ q'_\gamma(u) = -q_\gamma(u)\rho_\beta + q_\beta(\tau - u)\rho_\gamma \end{cases}$$

$$\begin{cases} q_\beta(u) = A\varrho_\beta + Be^{-\varrho_\beta\tau}e^{(\varrho_\beta - \varrho_\gamma)u} \\ q_\gamma(u) = A\varrho_\gamma + Be^{-\varrho_\gamma\tau}e^{(\varrho_\gamma - \varrho_\beta)u} \end{cases}$$

A and B determined with boundary and normalisation conditions.

CIS coincidence correction

- Particular case $r_\beta = r_\gamma$ and $\tau_\beta = \tau_\gamma$

$$A = \frac{1}{(1+\varrho_\beta\tau)(1+\varrho_\gamma\tau)}$$

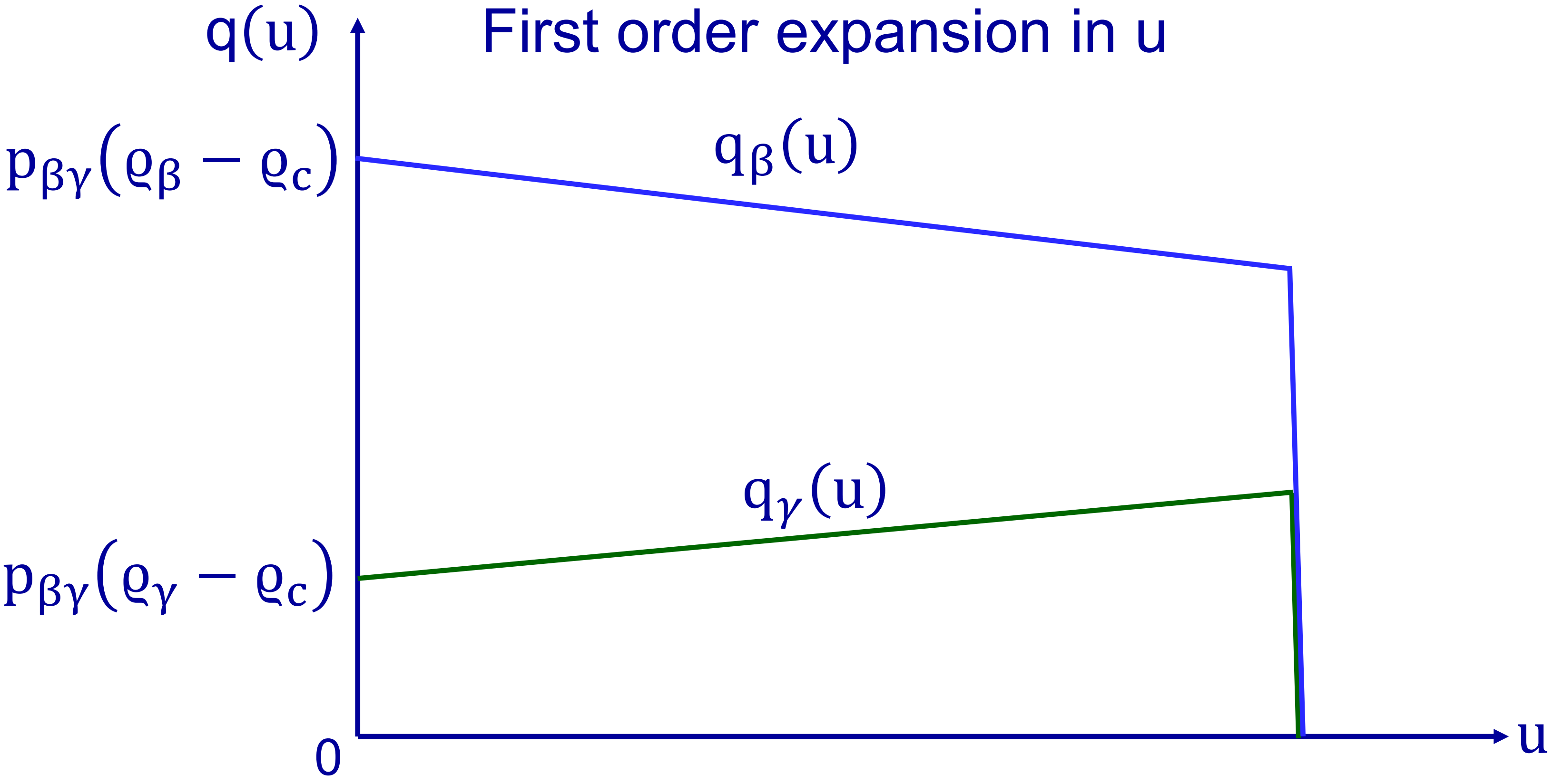
$$B = \frac{1}{(1+\varrho_\beta\tau)(1+\varrho_\gamma\tau)} \cdot \frac{\varrho_c(\varrho_\beta+\varrho_\gamma)e^{(\varrho_\beta-\varrho_\gamma)\tau}}{(\varrho_\beta-\varrho_c)e^{\varrho_\beta\tau}-(\varrho_\gamma-\varrho_c)e^{\varrho_\gamma\tau}}$$

$$p_{\beta\gamma} = \frac{1}{(1+\varrho_\beta\tau)(1+\varrho_\gamma\tau)} \cdot \frac{\varrho_\beta e^{\varrho_\beta\tau}-\varrho_\gamma e^{\varrho_\gamma\tau}}{(\varrho_\beta-\varrho_c)e^{\varrho_\beta\tau}-(\varrho_\gamma-\varrho_c)e^{\varrho_\gamma\tau}}$$

CIS coincidence correction

- Particular case $r_\beta = r_\gamma$ and $\tau_\beta = \tau_\gamma$

First order expansion in u



Bryant and Campion formulae are premised on constant $q_\beta(u)$ or $q_\gamma(u)$.

CIS coincidence correction

- Particular case $r_\beta = r_\gamma$ (=r) and $\tau_\beta = \tau_\gamma$

$$\rho_c = \frac{\widetilde{R}_c\left(\varrho_\beta e^{\varrho_\beta \tau} - \varrho_\gamma e^{\varrho_\gamma \tau}\right)}{\widetilde{R}_c\left(\varrho_\beta e^{\varrho_\beta \tau} - \varrho_\gamma e^{\varrho_\gamma \tau}\right) + p_\beta p_\gamma \left(\varrho_\beta e^{\varrho_\beta \tau} e^{(\varrho_\gamma - \varrho_\beta)r} - \varrho_\gamma e^{\varrho_\gamma \tau} e^{(\varrho_\beta - \varrho_\gamma)r}\right)}$$

$$\widetilde{R}_c = R_c - 2R_\beta R_\gamma r$$

$$p_\beta = \frac{1}{1 + \varrho_\beta \tau} \qquad p_\gamma = \frac{1}{1 + \varrho_\gamma \tau}$$

CIS coincidence correction

- Particular case $r_\beta = r_\gamma$ and $\tau_\beta = \tau_\gamma$

$$\rho_c = \frac{R_c - (r_\beta + r_\gamma)R_\beta R_\gamma}{(1 - R_\beta\tau_\beta)(1 - R_\gamma\tau_\gamma)\left\{1 + \frac{2R_c\tau_{min} - R_\gamma r_\beta - R_\beta r_\gamma}{2 - R_\beta r_\beta - R_\gamma r_\gamma}\right\}}$$

Bryant 1963

Bryant's formula obtains as a limit of CIS formula if $\rho_\beta = \rho_\gamma$.

For $\rho_\beta \neq \rho_\gamma$, Bryant's formula is a first order approximation (in $[\rho_\beta - \rho_\gamma]$) of CIS formula.

CIS coincidence correction

- General case $r_\beta \neq r_\gamma$ and $\tau_\beta > \tau_\gamma$ ($\tau_\beta = n \cdot \tau_\gamma$, n integer)

$$q_\gamma(u) = A_0 \frac{\varrho_\gamma}{\varrho_\beta} + \sum_{i=1}^{n} A_i \left(\frac{\varrho_\gamma + \varphi_i}{\varrho_\beta} \right) e^{-\varphi_i (u - \tau_\gamma)}$$

$$q_\beta(u) = A_0 + \sum_{i=1}^{n} A_i \left(\frac{\varrho_\gamma}{\varrho_\gamma + \varphi_i} \right)^k e^{-\varphi_i (u - k \cdot \tau_\gamma)}$$

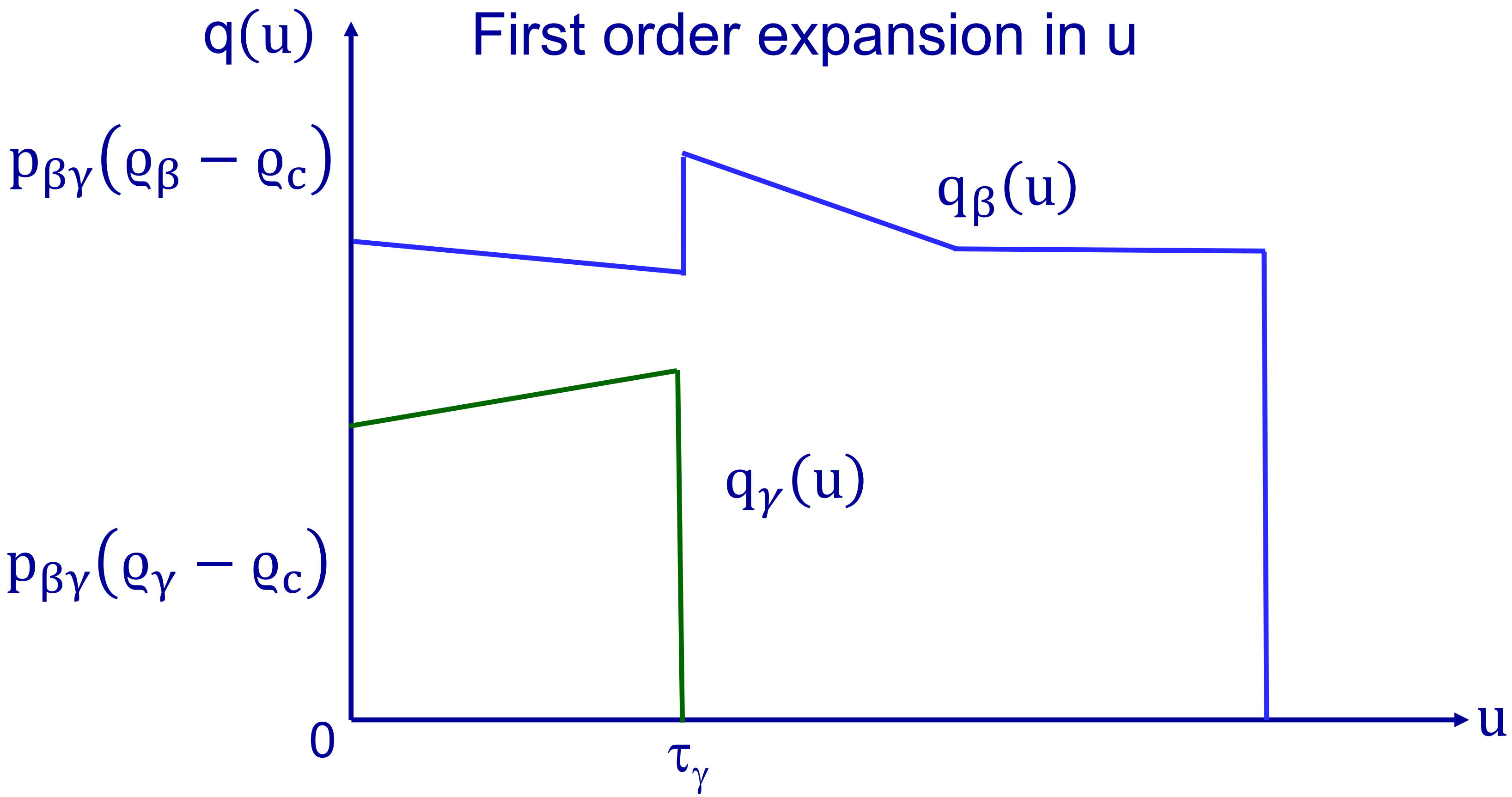
$$u \in \left[k\tau_\gamma, (k+1)\tau_\gamma \right] \qquad k = 0, 1, \ldots, n-1$$

φ_i are n roots of

$$(\varrho_\beta - \varphi)(\varrho_\gamma - \varphi)^n = \varrho_\beta \varrho_\gamma^n$$

CIS coincidence correction

- General case $r_\beta \neq r_\gamma$ and $\tau_\beta > \tau_\gamma$



Bryant and Campion formulae are premised on constant $q_\beta(u)$ or $q_\gamma(u)$.

CIS coincidence correction

- General case $r_\beta \neq r_\gamma$ and $\tau_\beta > \tau_\gamma$ ($\tau_\beta = s \cdot \tau_\gamma$, s real)

$$q_\gamma(u) = p_{\beta\gamma}\left(\varrho_\beta - \varrho_c\right) + B_1 u + B_2 u^2 + B_3 u^3$$

$$q_\beta(u) = \begin{cases} p_{\beta\gamma}\left(\varrho_\beta - \varrho_c\right) + C_{10} u + C_{20} u^2 & \text{if } u \leq \tau_\gamma \\ q_\beta\left(k\tau_\gamma - \delta\tau\right) + C_{1k}\left(u - k\tau_\gamma\right) + C_{20}\left(u - k\tau_\gamma\right)^2 & \end{cases}$$

$$\text{if } u \in \left[k\tau_\gamma, (k+1)\tau_\gamma\right] \quad k = 0, 1, \ldots, s-1$$

CIS coincidence correction

- General case $r_\beta \neq r_\gamma$ and $\tau_\beta > \tau_\gamma$ ($\tau_\beta = s \cdot \tau_\gamma$, s real)

$$\rho_c = \frac{R_c - (r_\beta + r_\gamma)R_\beta R_\gamma}{(1 - R_\beta \tau_\beta)(1 - R_\gamma \tau_\gamma) \cdot X(r_\beta, r_\gamma) + R_c \tau_\gamma \cdot Y}$$

$$X(r_\beta, r_\gamma) = e^{-\varrho_\gamma r_\beta} + e^{-\varrho_\beta r_\gamma} - 1 - \varrho_\beta \varrho_\gamma \cdot \omega(r_\beta, r_\gamma)$$

$$\begin{aligned} \omega(r_\beta, r_\gamma) = & \frac{r_\beta^2}{2!}\{1 - \tau_\gamma(\varrho_\beta - \delta(s-1)\varrho_\gamma)\} + \\ & \frac{r_\gamma^2}{2!}\{\delta(s-1) + \tau_\gamma(\varrho_\beta - \delta(s-1)\varrho_\gamma)\} - \\ & \frac{r_\beta^3}{3!}\{\varrho_\gamma(1 + \delta(s-1)) - \varrho_\beta\} - \\ & \frac{r_\gamma^3}{3!}\{2\varrho_\beta \delta(s-1) + \varrho_\gamma(\delta(s-2) - \delta(s-1))\} \end{aligned}$$

CIS coincidence correction

- General case $r_\beta \neq r_\gamma$ and $\tau_\beta > \tau_\gamma$ ($\tau_\beta = s\,\tau_\gamma$, s real)

$$Y = 1 - Y_1 \frac{\tau_\gamma}{2!} + Y_2 \frac{\tau_\gamma^2}{3!} - Y_3 \frac{\tau_\gamma^3}{4!} + \ldots$$

$$Y_1 = \varrho_\beta + \{2 - \delta(s-1)\varrho_\gamma\}$$

$$Y_2 = \varrho_\beta^2 + \{6 - 2s\delta(s-1)\}\varrho_\beta\varrho_\gamma + \{6 - \delta(s-2) - 5\delta(s-1)\}\varrho_\gamma^2$$

$$Y_3 = \varrho_\beta^3 + \{14 - 3s\delta(s-1)\}\varrho_\beta^2\varrho_\gamma + \{36 - 2\delta(s-2) - 25\delta(s-1)\}\varrho_\beta\varrho_\gamma^2 + \{12 - \delta(s-3) - 6\delta(s-2) - 11\delta(s-1)\}\varrho_\gamma^3$$

CIS coincidence correction

- General case $r_\beta \neq r_\gamma$ and $\tau_\beta > \tau_\gamma$ (τ_β = s τ_γ, s real)

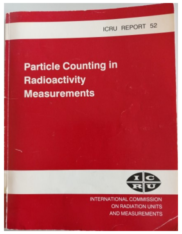
$$s = \frac{\tau_\beta}{\tau_\gamma} \qquad \delta(z) = \begin{cases} 1 - |z|, & 0 \le |z| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

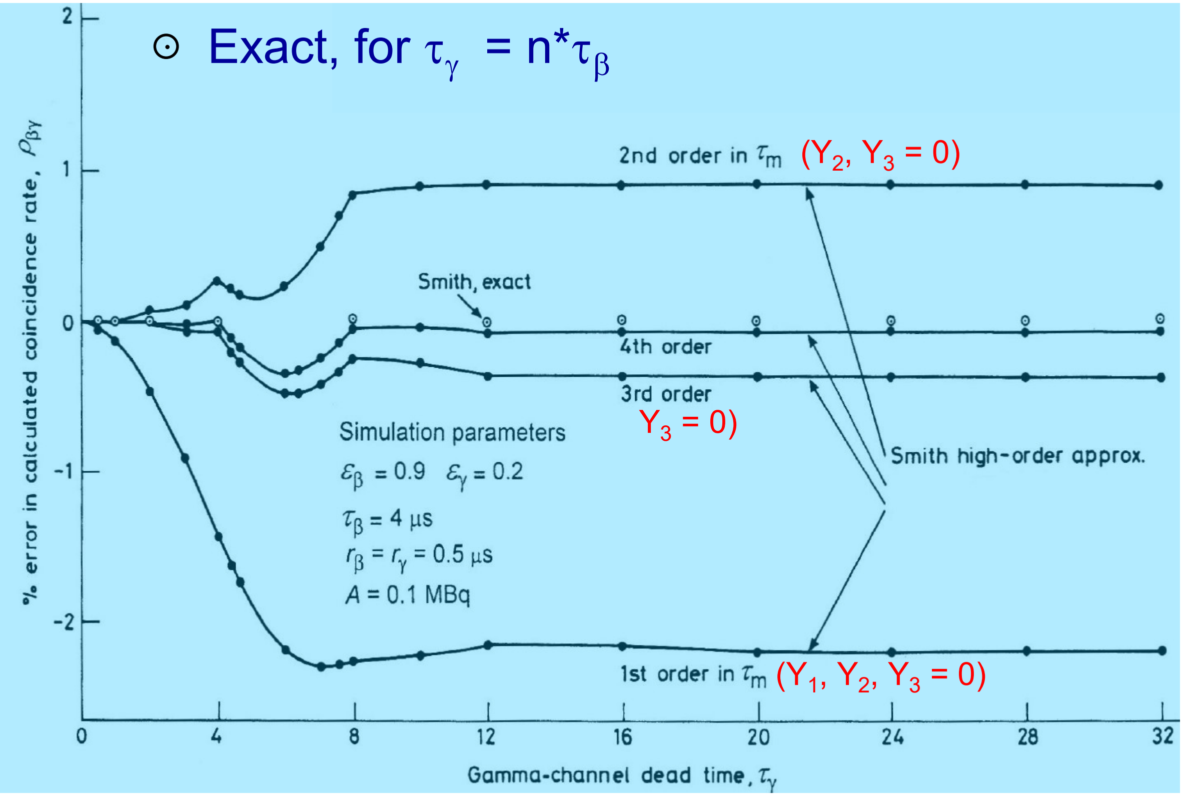
4th order in τ and 3rd order in r.

CIS coincidence correction

- Higher-order approximation with increasing τ_γ

  Monte Carlo simulated data





ρ_c estimates converge quickly with Y terms.

β-γ coincidence counting

$$\rho_c = \frac{R_c - (r_\beta + r_\gamma) R_\beta R_\gamma}{(1 - R_\beta \tau_\beta - R_\gamma \tau_\gamma + R_c \tau_{min})(1 - R_\gamma r_\beta - R_\beta r_\gamma)}$$

$$\tau_{min} = \min(\tau_\beta, \tau_\gamma)$$

Campion 1959

$$\rho_c = \frac{R_c - (r_\beta + r_\gamma) R_\beta R_\gamma}{\frac{1}{1 + \varrho_\beta r_\beta + \rho_\gamma r_\gamma - \varrho_c \tau_{min}} - R_\gamma (1 - R_\beta \tau_\beta) r_\beta - R_\beta (1 - R_\gamma \tau_\gamma) r_\gamma}$$

Gandy 1962

β-γ coincidence counting

$$\rho_c = \frac{R_c - (r_\beta + r_\gamma)R_\beta R_\gamma}{(1 - R_\beta \tau_\beta)(1 - R_\gamma \tau_\gamma)\left\{1 + \frac{2R_c\tau_{min} - R_\gamma r_\beta - R_\beta r_\gamma}{2 - R_\beta r_\beta - R_\gamma r_\gamma}\right\}}$$

Bryant 1963

$$\rho_c = \frac{R_c - (r_\beta + r_\gamma)R_\beta R_\gamma}{(1 - R_\beta \tau_\beta)(1 - R_\gamma \tau_\gamma)X + R_c\tau_{min}Y}$$

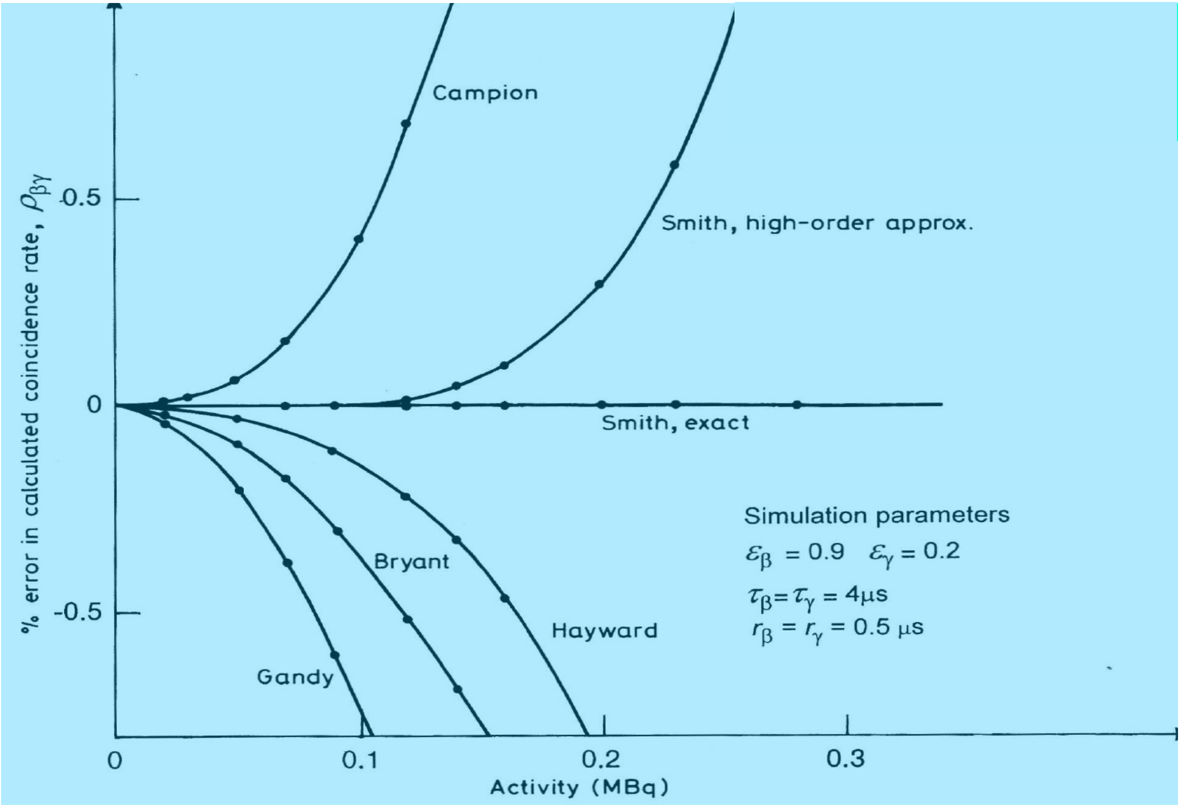
CIS, 1977-8

$$X \equiv X(\rho, r, \tau) \qquad Y \equiv Y(\rho, \tau)$$

4th order in τ and 3rd order in r.

CIS coincidence correction

- Comparison with alternative prescriptions $\tau_\beta = \tau_\gamma$
  Monte Carlo simulated data

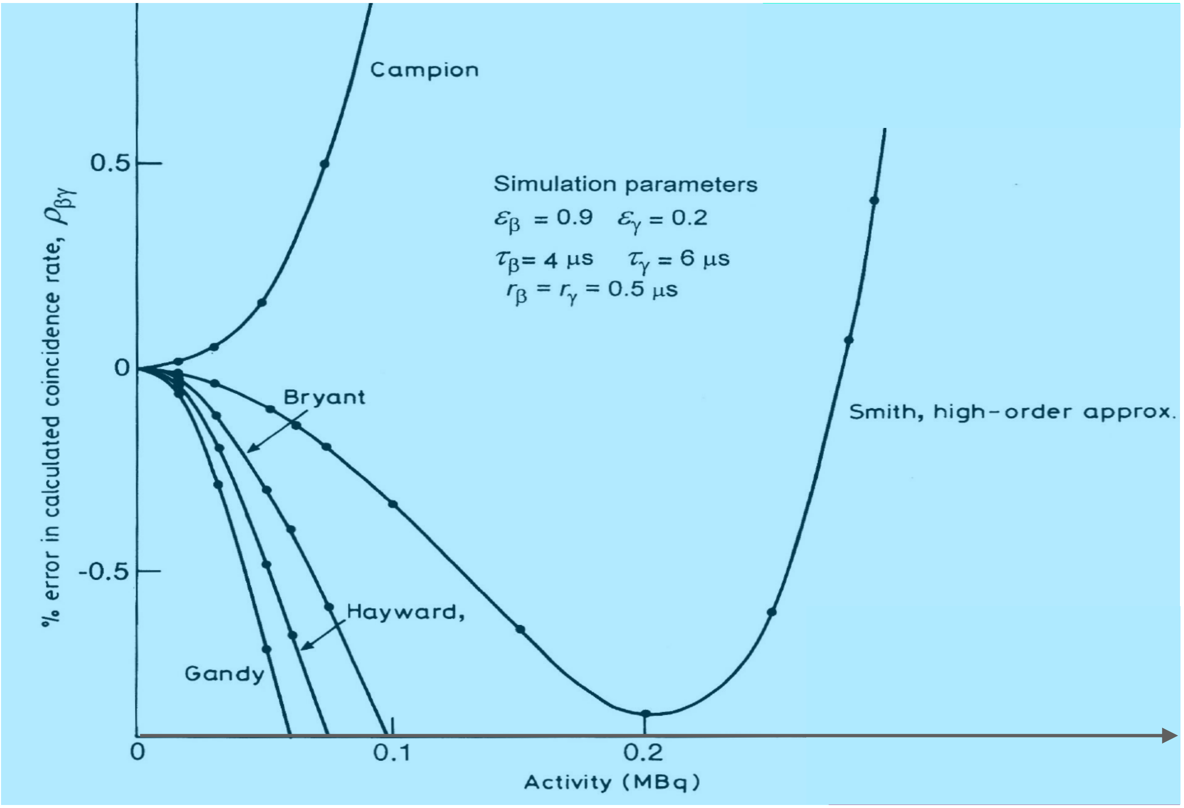


Higher order CIS approximation breaks down as $\rho\tau \rightarrow 1$.

CIS coincidence correction

- Comparison with alternative prescriptions $\tau_\gamma = 1.5*\tau_\beta$

  Monte Carlo simulated data

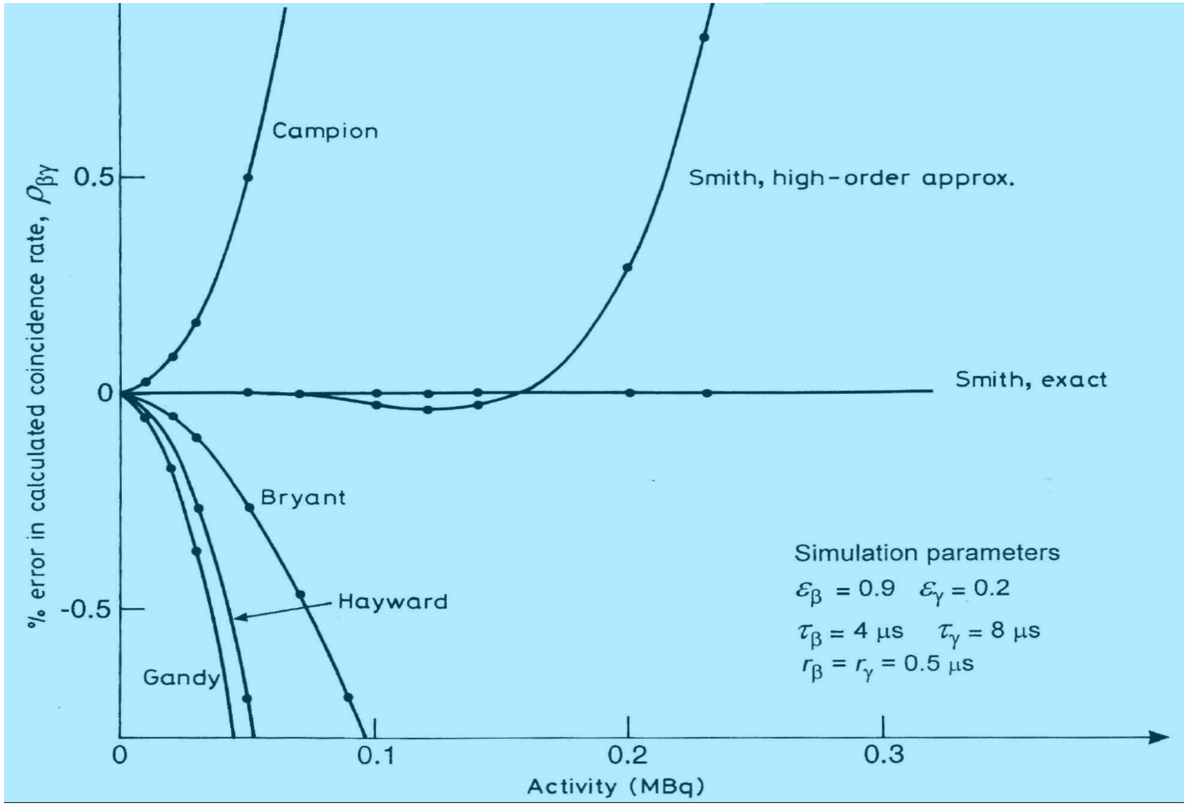


Higher order CIS approximation breaks down as $\rho\tau \to 1$.

CIS coincidence correction

- Comparison with alternative prescriptions $\tau_\gamma = 2^*\tau_\beta$

  Monte Carlo simulated data

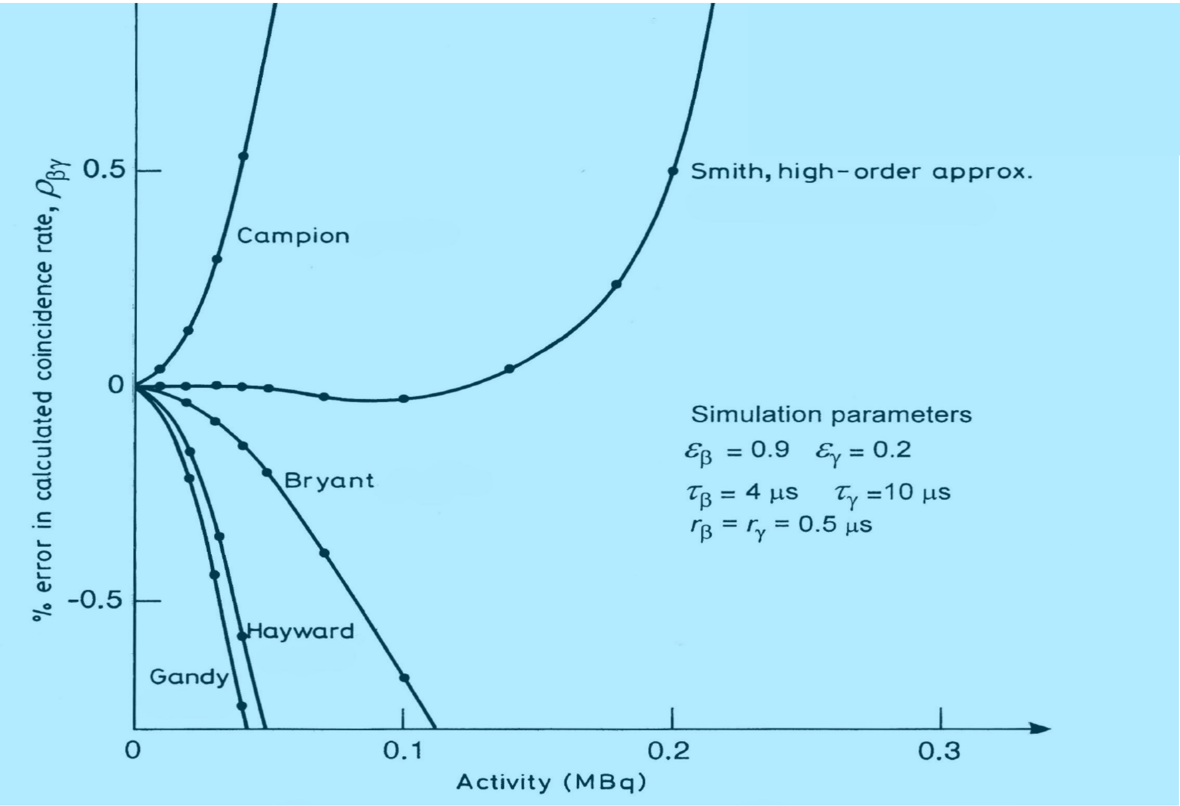


Higher order CIS approximation breaks down as $\rho\tau \rightarrow 1$.

CIS coincidence correction

- Comparison with alternative prescriptions $\tau_\gamma = 2.5^*\tau_\beta$

  Monte Carlo simulated data

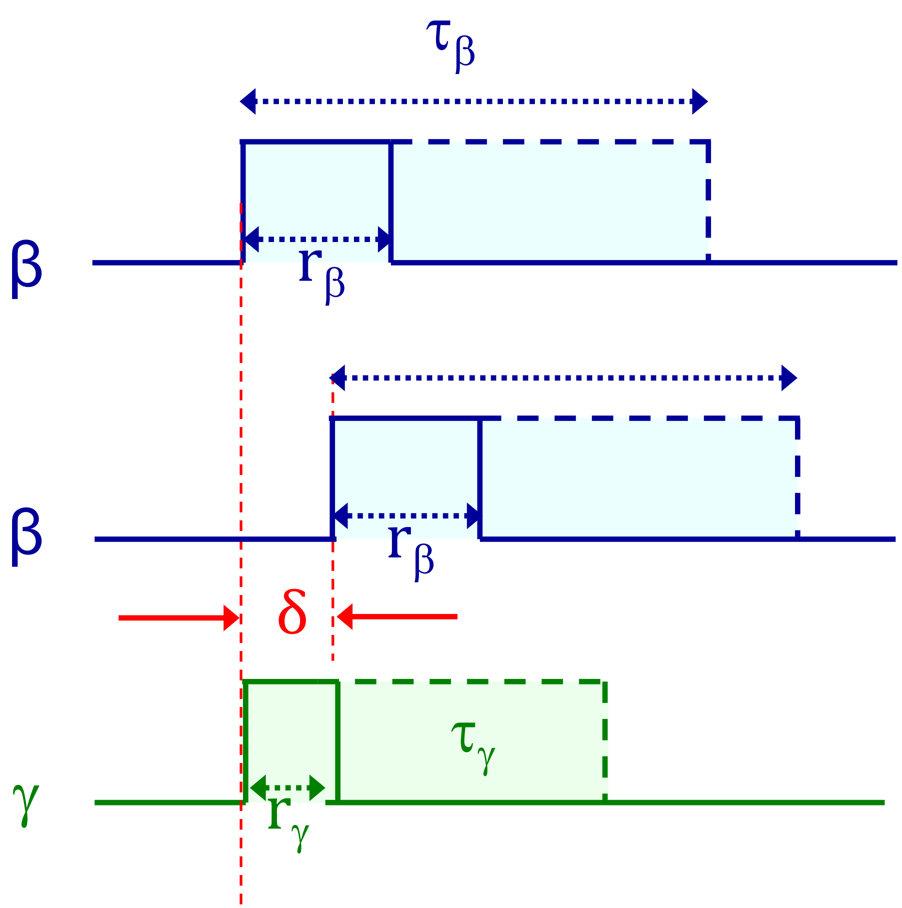


Higher order CIS approximation breaks down as $\rho\tau \rightarrow 1$.

Fixed β-γ delay

- Fixed delay between β- and γ-pulses

Beta pulses are delayed by a fixed delay $\delta \leq r_\beta$.

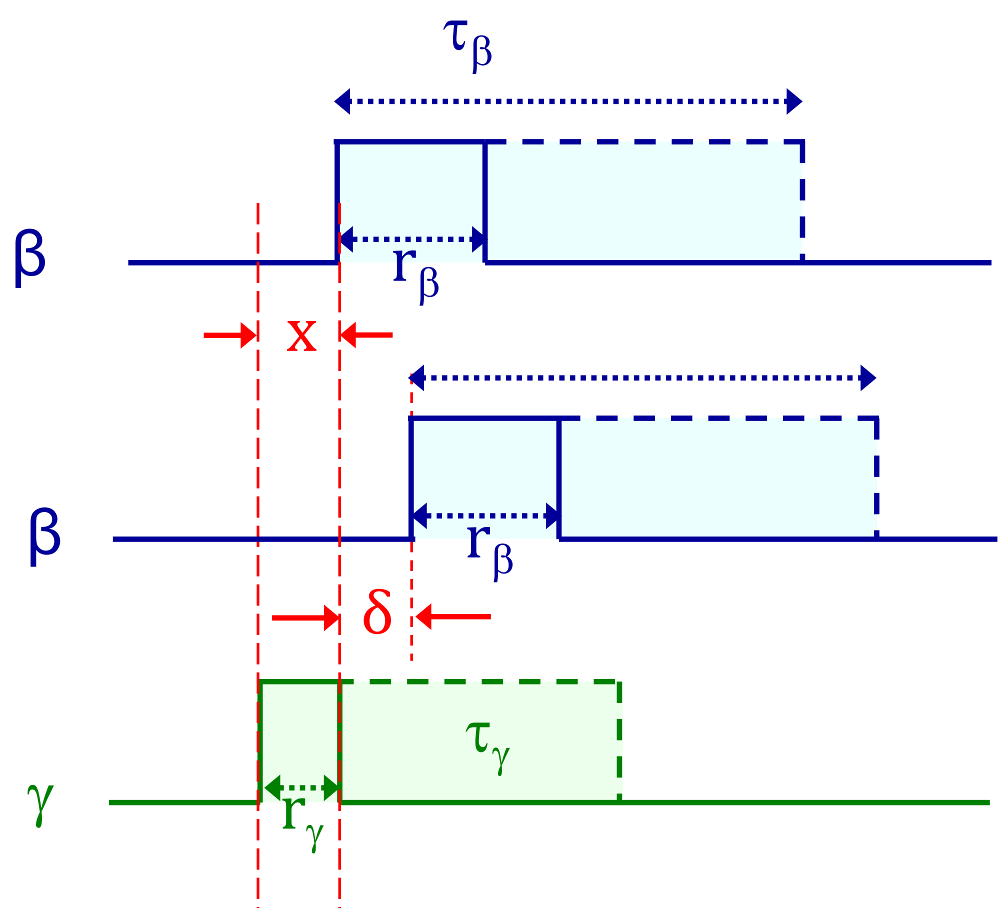


Number of genuine coincidences is not affected

Fixed β-γ delay

- Fixed delay between β- and γ-pulses

Beta pulses are delayed by a fixed delay $\delta \leq r_\beta$.



Number of fortuitous coincidences is altered

$0 \leq x \leq r_\gamma$

Fixed β-γ delay

- Fixed delay between β- and γ-pulses

  Formalism for calculating the observed coincidence rate is the same except that

$$r_\beta \rightarrow r'_\beta = r_\beta + \delta$$

Effective resolving time

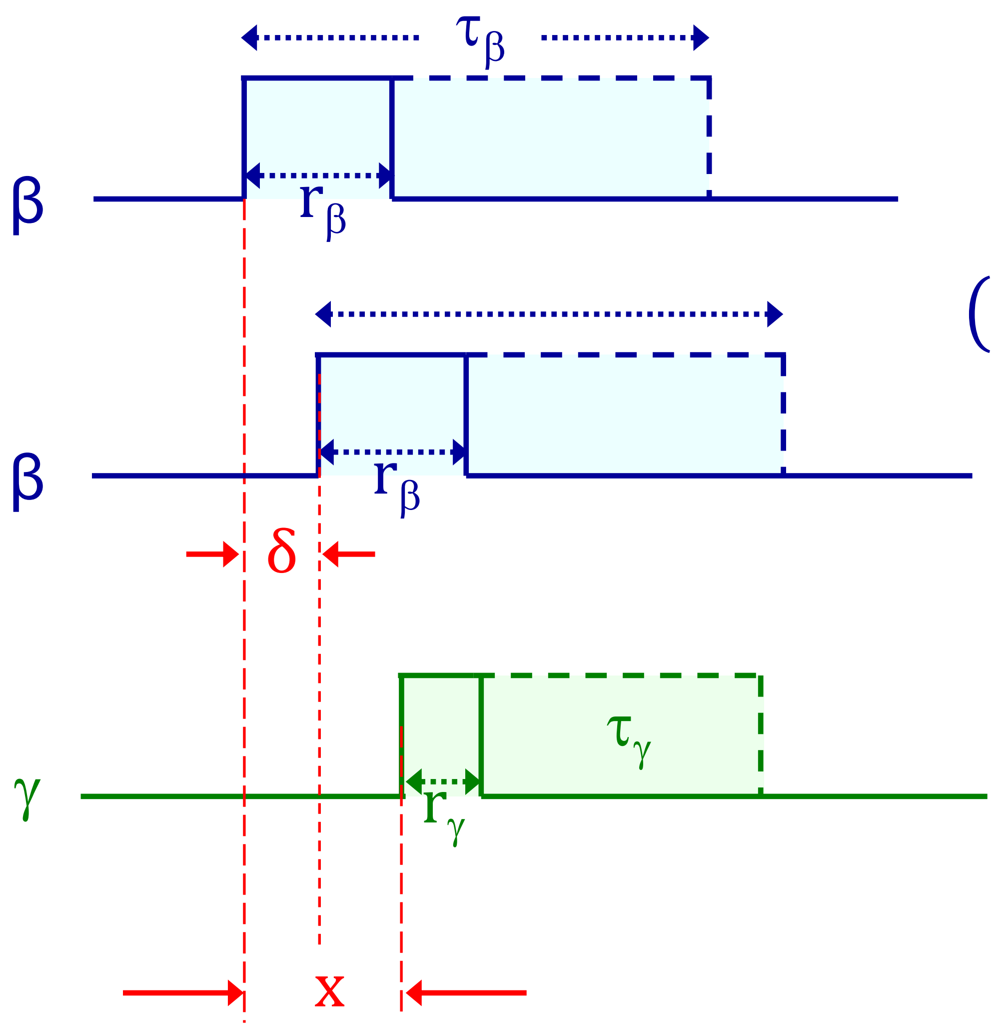
$$r_\gamma \rightarrow r'_\gamma = r_\gamma - \delta$$

in the limits of the integrals, and the approximations.

$$r_\beta + r_\gamma = r'_\beta + r'_\gamma$$

Fixed β-γ delay

- Type 1 fortuitous coincidences



β-first

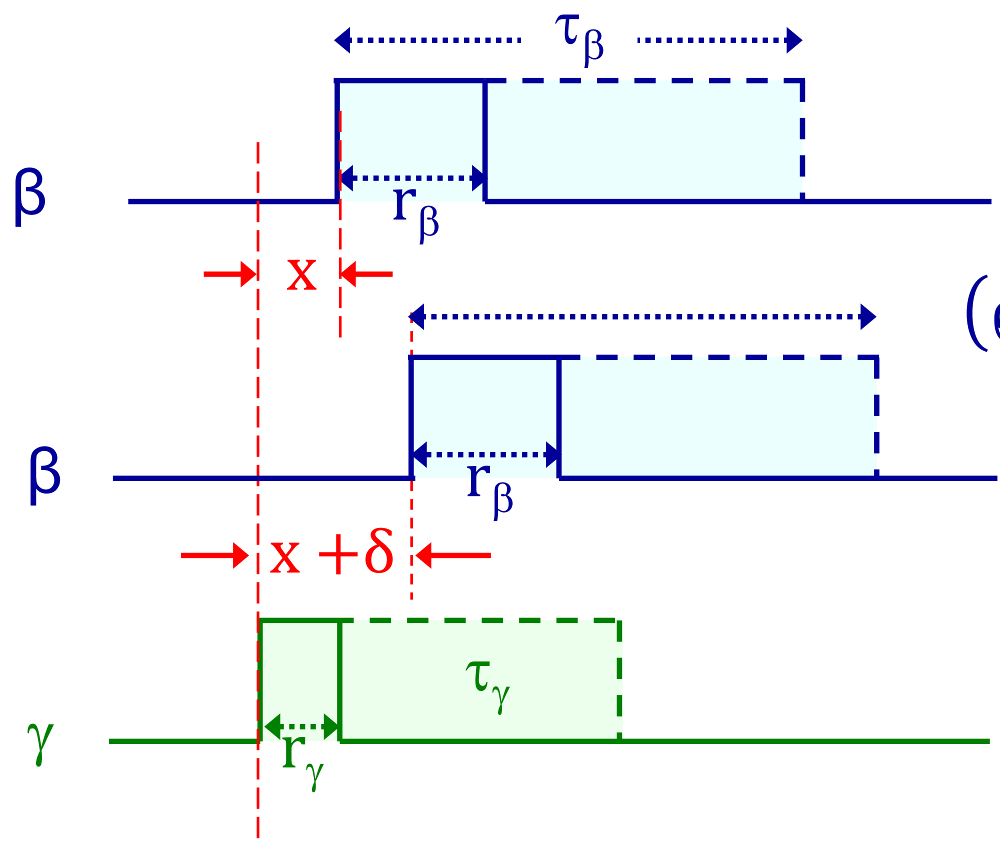
$$\left(\varrho_\beta - \varrho_c\right) \int_0^{r_\beta} e^{-\varrho_\gamma x} \varrho_\gamma dx$$

$$0 \leq x - \delta \leq r_\beta$$

$$0 \leq x \leq r_\beta + \delta$$

Fixed β-γ delay

- Type 1 fortuitous coincidences



γ-first

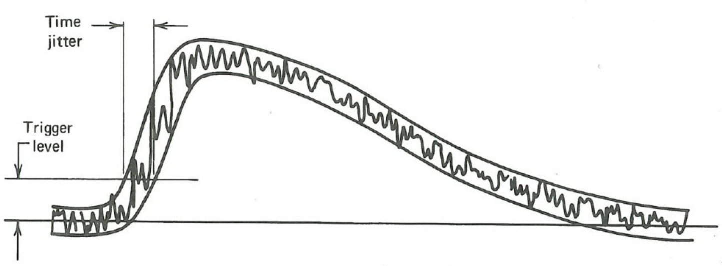
$$(\varrho_\gamma - \varrho_c)\int_0^{r_\gamma} e^{-\varrho_\beta x}\varrho_\beta dx$$

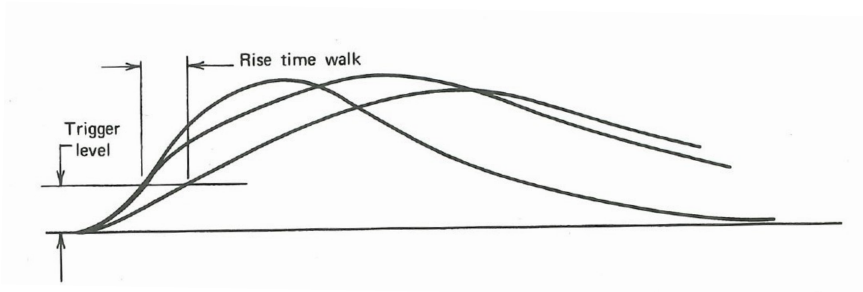
$$0 \leq x + \delta \leq r_\gamma$$
$$0 \leq x \leq r_\gamma - \delta$$

Time-jitter effect

- Time-jitter effect on the coincidence counting rate

In leading edge triggering, random fluctuations, changes in rise time or pulse shape cause variations in pulse timing.





Occurs in both channels but may regarded as a variable relative delay between β- and γ-channels.

Beta- and gamma-countrates are unaffected but coincidence rate is affected by time-jitter.

Time-jitter effect

- Time-jitter effect on the coincidence counting rate

Accidental coincidences

$$r_\beta \rightarrow r'_\beta = r_\beta + \bar{\delta}$$

$$r_\gamma \rightarrow r'_\gamma = r_\gamma - \bar{\delta}$$

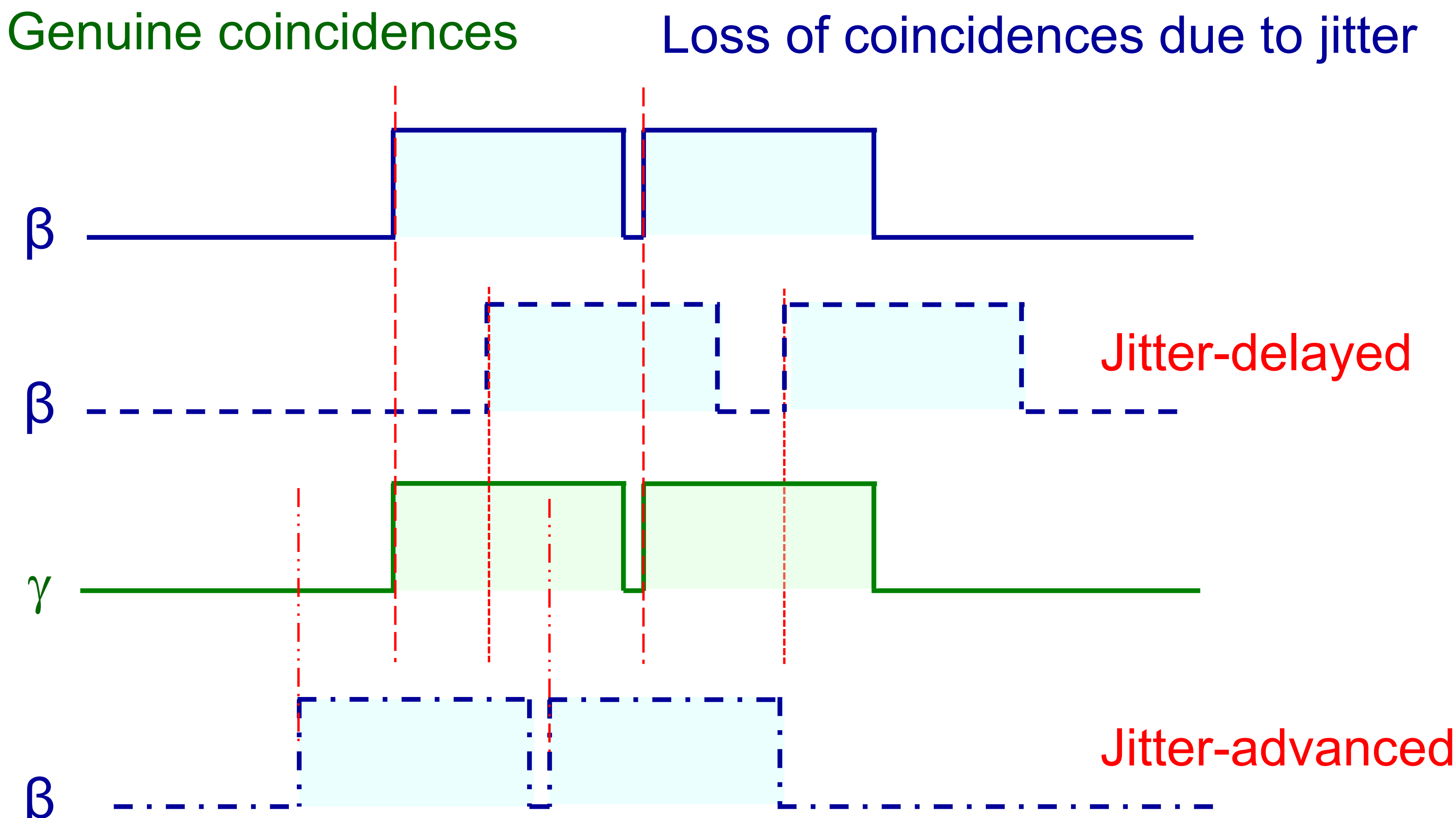
$$\bar{\delta} = \int_{\delta_{min}}^{\delta_{max}} \delta f(\delta) d\delta$$

In reality, time-jitter effect is more complex than simply averaging over a single relative delay distribution.

Monte-Carlo simulations show that except in the $\tau_\beta = \tau_\gamma$ case, effect of time jitter on accidental coincidence rate is very small.

Time-jitter effect

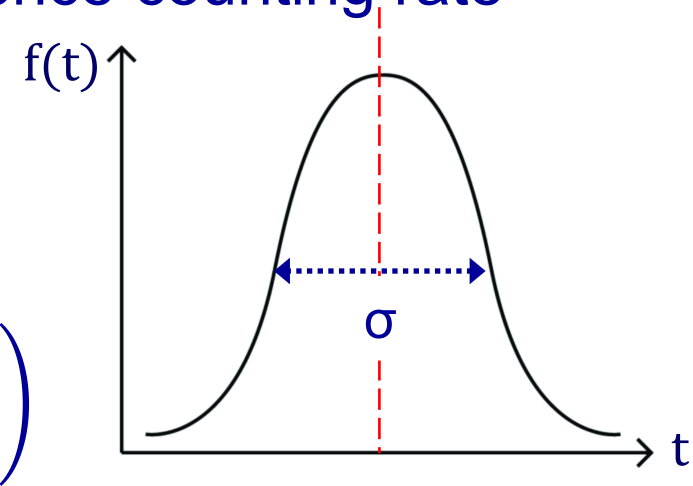
- Time-jitter effect on the coincidence counting rate



Jitter advance does not compensate jitter delay

Time-jitter effect

- Time-jitter effect on the coincidence counting rate



Genuine coincidences

$$R_{c_{\mathrm{Jitter-loss}}} = p_{\beta\gamma}\varrho_c \cdot \left(\rho_c \cdot \frac{\sigma}{\sqrt{3}} \cdot \zeta\right)$$

Measures the variability of time-jitter from the mean

$$\zeta = \begin{cases} 1, & \tau_\beta = \tau_\gamma\,, \\ 0 & |\tau_\beta - \tau_\gamma| > \sigma. \end{cases}$$

Time-jitter effect

- Fixed delay and time-jitter effects

$$\rho_c = \frac{\widetilde{R_c} - (r'_\beta + r'_\gamma) R_\beta R_\gamma}{(1 - R_\beta \tau_\beta)(1 - R_\gamma \tau_\gamma) \cdot X(r'_\beta, r'_\gamma) + \widetilde{R_c} \tau_\gamma \cdot Y}$$

$$r_\beta \to r'_\beta = r_\beta + \bar{\delta}$$

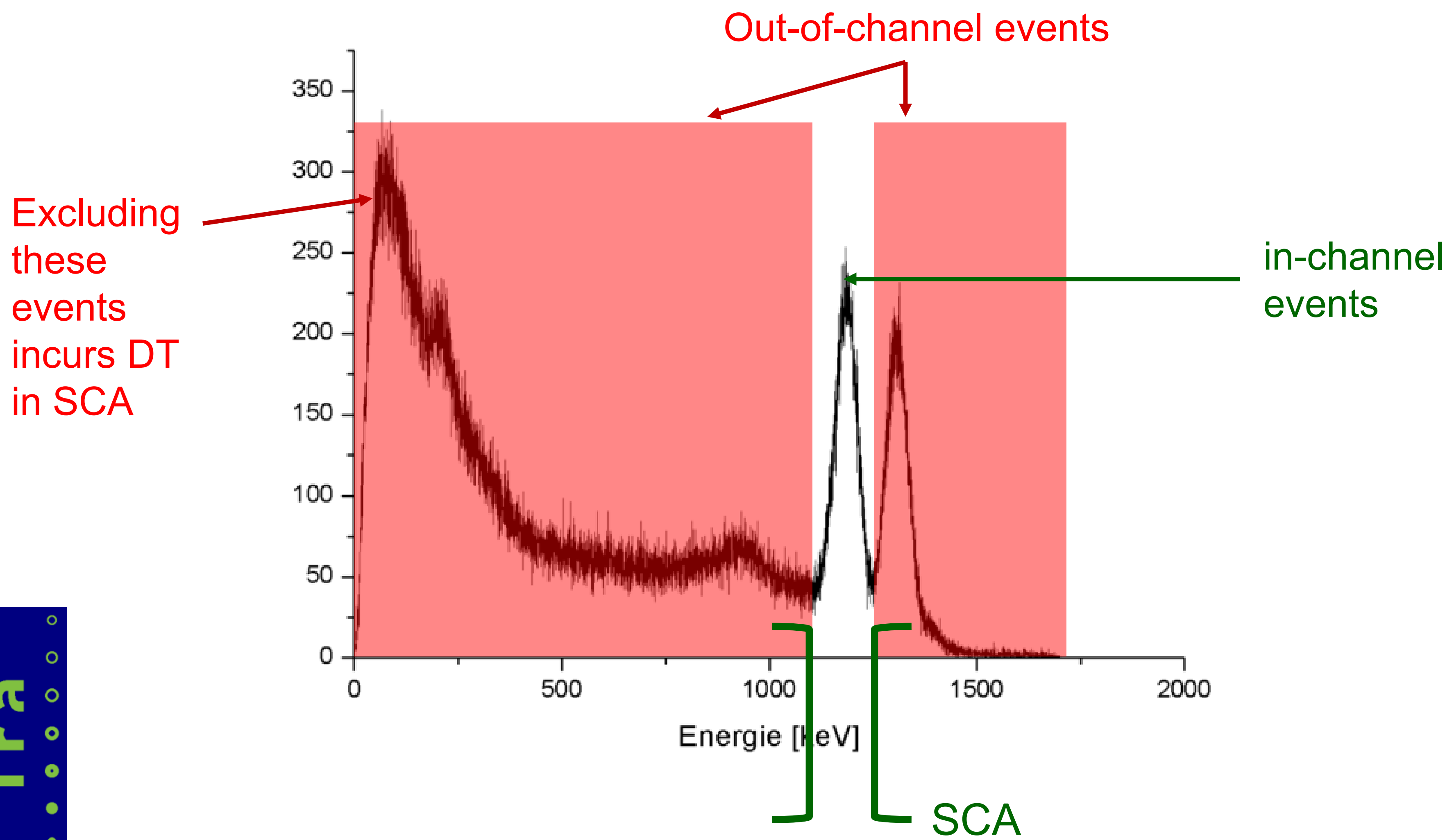
$$r_\gamma \to r'_\gamma = r_\gamma - \bar{\delta}$$

$$\widetilde{R_c} = R_c + p_{\beta\gamma} \varrho_c \cdot \left(\rho_c \cdot \frac{\sigma}{\sqrt{3}} \cdot \zeta \right)$$

$$\zeta = \begin{cases} 1, & \tau_\beta = \tau_\gamma \,, \\ 0 & |\tau_\beta - \tau_\gamma| > \sigma. \end{cases}$$

Out-of-channel events

- Taking into account out-of-channel events



Out-of-channel events

- General case $r_\beta = r_\gamma = r$ and $\tau_\beta > \tau_\gamma$ ($\tau_\beta = s\,\tau_\gamma$, s real)

$$\rho_{c_{in}} = \frac{R_{c_{in}} - 2rR_\beta R_{\gamma_{in}}}{\left(1 - R_\beta \tau_\beta\right)\left(1 - R_{\gamma_{in}}\tau_\gamma\right)\cdot X(r) + R_{c_{in}}\tau_\gamma \cdot \alpha \cdot Y}$$

$$X(r) \simeq 1 - r\left(\varrho_\beta + \alpha \cdot \varrho_{\gamma_{in}}\right) + \varphi(r)$$

$$\varphi(r) = \frac{r^2}{2!}\left\{\varrho_\beta^2 + \alpha \cdot \varrho_\gamma\, \varrho_{\gamma_{in}} - \alpha \cdot \varrho_\beta\, \varrho_{\gamma_{in}}\left(1 + \delta(s-1)\right)\right\} -$$

$$\frac{r^3}{3!}\left\{\varrho_\beta^3 + \alpha \cdot \varrho_\gamma^2 \varrho_{\gamma_{in}} - \right.$$

$$\left.\alpha \cdot \varrho_\beta \varrho_{\gamma_{in}}\left[\varrho_\beta\left(2\delta(s-1) - 1\right) + \varrho_\gamma\left(\delta(s-2) + 1\right)\right]\right\}$$

$$\alpha = \frac{\varrho_\gamma}{\varrho_{\gamma_{in}}}$$

Computer codes

$$Y = 1 - Y_1 \frac{\tau_\gamma}{2!} + Y_2 \frac{\tau_\gamma^2}{3!} - Y_3 \frac{\tau_\gamma^3}{4!} + \ldots$$

$$Y_1 = \varrho_\beta + \{2 - \delta(s-1)\varrho_\gamma\}$$

$$Y_2 = \varrho_\beta^2 + \{6 - 2s\delta(s-1)\}\varrho_\beta\varrho_\gamma + \{6 - \delta(s-2) - 5\delta(s-1)\}\varrho_\gamma^2$$

$$Y_3 = \varrho_\beta^3 + \{14 - 3s\delta(s-1)\}\varrho_\beta^2\varrho_\gamma + \{36 - 2\delta(s-2) - 25\delta(s-1)\}\varrho_\beta\varrho_\gamma^2 + \{12 - \delta(s-3) - 6\delta(s-2) - 11\delta(s-1)\}\varrho_\gamma^3$$

Out-of-channel events

- General case $r_\beta = r_\gamma = r$ and $\tau_\beta > \tau_\gamma$ ($\tau_\beta = s\,\tau_\gamma$, s real)

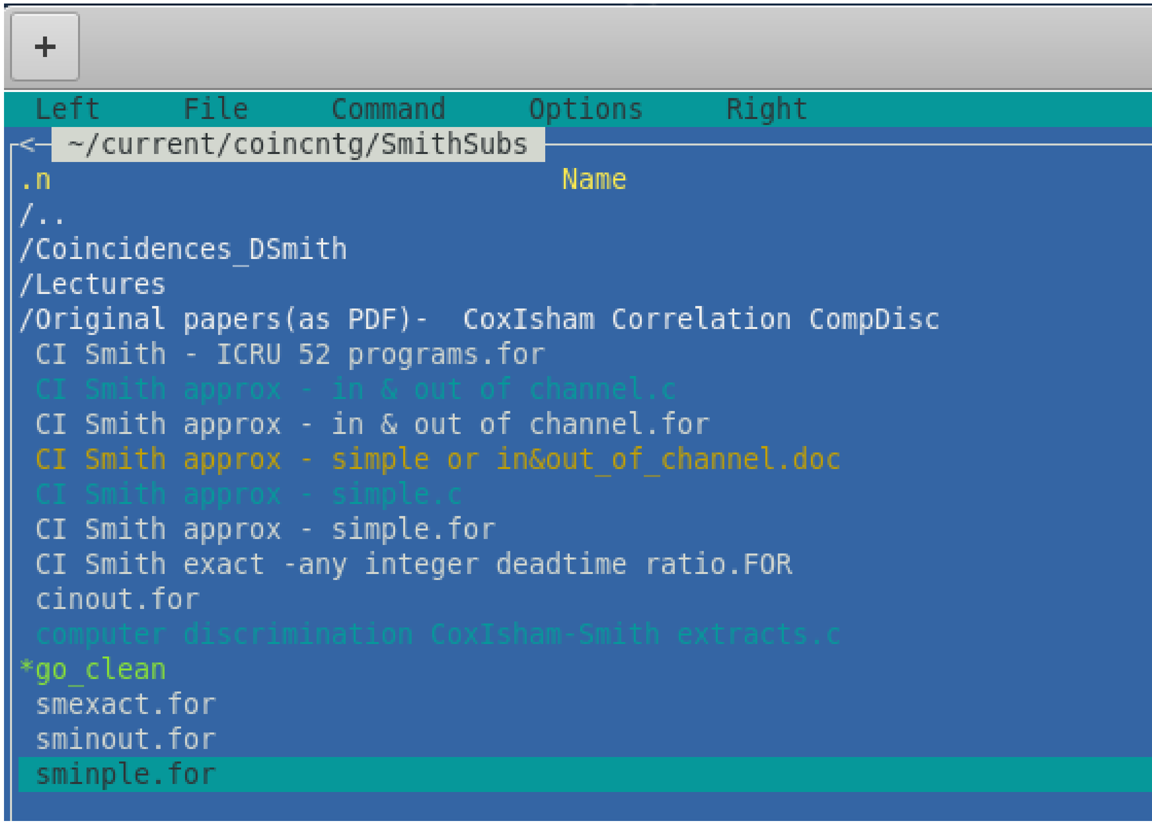
$$\rho_{c_{in}} \simeq \frac{R_{c_{in}} - 2rR_\beta R_{\gamma_{in}} + \rho_c\left(p_\beta p_\gamma \rho_{\gamma_{in}} r - R_{c_{in}}\tau_{out}\right)}{p_\beta p_\gamma e^{-\rho_\beta r} + R_{c_{in}}\left(\tau_\gamma - \tau_{out}\right)}$$

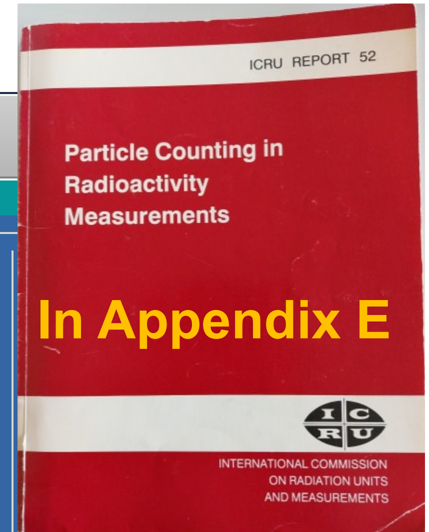
$$p_\beta = \frac{1}{1 + \varrho_\beta \tau_\beta} \qquad p_\gamma = \frac{1}{1 + \varrho_{\gamma_{in}}\tau_\gamma + \left(\rho_\gamma - \rho_{\gamma_{in}}\right)\tau_{out}}$$

$$\tau_{out} = \min\left(\tau_\beta, \tau_{\gamma_{out}}\right)$$

Computer codes

- Codes that implement CIS correction





Computer codes

- Codes that implement CIS correction (Fortran)

$\tau_\beta = n \cdot \tau_\gamma$, n integer

```
CI Smith exact -any integer deadtime ratio.FOR - GNU Emacs at hos64748
File  Edit  Options  Buffers  Tools  Fortran  Help
Save  Undo

         PROGRAM  SMITHEXACT

C  Comment added in February 2007
C    uses various NAG routines, which the user will have to supply :
C    to solve polynomial               by  CALL C02AEF(A,N,REZ,IMZ,TOL,IFAIL)
C    to solve simultaneous equations   by  CALL F04ADF(B,31,Y,31,R+2,1,C,31,WKSPCE,IFAIL)
C    to x such that supplied f(x)=0    by  CALL C05ADF(Z/1000,Z,Z/1E9,0.0,CRESIDUAL,L12,IFAIL)
C
C    use GOOGLE internet search (eg google "NAG F04ADF") to find information about these NAG routines
C
C    This Fortran program was last used in about 1988.  Since then, the high-order approximation
C             of Cox-Isham/Smith was used instead of this "exact" Cox-Isham/Smith solution.
C             The exact solution only applies when one dead time is an integer multiple of the other.
C
C             The high-order approximation (which is 4th order in deadtime and 3rd order in
C             resolving time) is trivial to use,valid for all dead times (ie not resticted to
C             integer deadtime ratios) and is almost as accurate as the exact
C             solution, provided (count rate)*(smallest deadtime) does NOT approach 1.0.
C
C
C     INPUT DATA =  5 COUNT RATES AND THE DEADTIMES AND RESOLVING TIMES.
C     PROGRAM CALLS EXACT SOLUTION IN CALL EXACTSMITH,
C                     THEN CALL DSMITHAPPRINOUT   = HIGH-ORDER-APPROX FOR 5 SCALERS (WITH OUT-OF-CHANNELS)
C                     THEN CALL DSMITHAPPR        = HIGH ORDER-APPROX FOR 3 SCALERS (NO OUT-OF-CHANNELS)
C
C     PROGRAM THEN EVALUATES VARIOUS FIRST-ORDER FORMULAE, SUCH AS cAMPION ETC
C
C     PPROGRAM USED TO OBTAIN EXACT CORRECTION, BUT ALSO TO COMPARE THE EXACT WITH HIGH-ORDER APPROXIMATION
C                AND WITH CAMPION ETC.
C
C  END OF FEBRUARY 2007 COMMENT
C
C   ---------------------------------------------------------------------------------------------------------
C
-(DOS)---  CI Smith exact -any integer deadtime ratio.FOR   Top   L1    (Fortran)
For information about GNU Emacs and the GNU system, type C-h C-a.
```

Computer codes

- Codes that implement CIS correction (Fortran, C)

$\tau_\beta = s \cdot \tau_\gamma$, s non-integer

CI Smith approx - simple.for - GNU Emacs at hos64748

File Edit Options Buffers Tools Fortran Help

```fortran
      SUBROUTINE DSMITHAPPR(B,G,C,   TB,TG,   HB,HG,   RB,RG,L12)
C
C     Implements simple COX-ISHAM/SMITH solution, for observed
C     count rates B,G C.
C     IMPLEMENTS EQU. 5.51 of ICRU 52, WITH Y and X OF ICRU 52 DEFINED
C     BY  EQU. 5.55  AND 5.56 OF ICRU 52.
C      The approximation is valid for any deadtime & resolving time
C      values, subject to the usual limitations :
C           - the safe limitation is  (HB+HG) < minimum(TB,TG)
C           - resolving times must be set to properly cover the real
C                time-distribution.
C               (for details see ICRU 52).
C
C      Written in FORTRAN
C
C
C     INPUT
C       B          observed beta count rate
C       G,C        observed gamma and coincidence count rates
C       TB,TG      DEAD TIMES
C       HB,HG      EFFECTIVE RESOLVING TIMES FOR B AND G IN
C                                       THE MEASUREMENT OF C.
C
C     OUTPUT
C       RB        CORRECTED B
C       RG        CORRECTED G
C       L12       CORRECTED C (APPROX)
C       the following 4 parameters are evaluated in the subroutine, and
C       could be output for checking or research:
C         CGEN    CALC. OBSERVED  GENUINE COINCS. IN C. (CGEN=P12*L12)
C         CACC    CALC. OBSERVED  ACCID. COINCS.  IN   C
C         CACC2   CALC. OBSERVED 2ND.TYPE ACCID.COINCS. IN C
C            ( C= CGEN+CACC.    CACC2 IS PART OF CACC)
C         P12      CALCULATED BY 5.54 of ICRU 52
C
C
C    HB AND HG MAY BE UNEQUAL IN REALITY  AND/OR EFFECTIVELY
C    UNEQUAL DUE TO relative Beta-Gamma DELAY at the coincidence unit(s).
C     I.E.
```

-(DOS)--- CI Smith approx - simple.for Top L1 (Fortran)

CI Smith approx - simple.c - GNU Emacs at hos64748

File Edit Options Buffers Tools C Help

```c
/* cox-isham/smith  correction
   In principle, this is identical to ICRU 52, appendix E (first part).
   The ICRU 52 version in in FORTRAN 77.
   This version here is in "C", written in the same notation as ICRU 52.

  The approximation is valid for any deadtime and resolving time values,
  subject to the usual limitations :
   - the safe limitation is  (rb+rg) < minimum(tb,tg)
   - resolving times must be set to properly cover the real time-distribution
      (for details see ICRU 52)

  Bobs, Gobs, Cobs    = the observed data in the 3 counters (counts per second)
  tb, tg              = dead times
  rb, rg              = resolving times
  Btrue,Gtrue,Ctrue   = the fully corrected results (counts per second)

*/

void dsmith_approximation(
  double  Bobs,   double   Gobs,  double Cobs,
  double  tb,     double   tg,
  double  rb,     double   rg,    double delayB,
  double *Btrue,  double *Gtrue,  double *Ctrue)

 {double pb,pg,s,MAX,MIN,tmin,rmax,rmin,X,Y;
  pb=1-Bobs*tb;     pg=1-Gobs*tg;     *Btrue=Bobs/pb;     *Gtrue=Gobs/pg;
  if (tb >= tg)
       {s=tb/tg; tmin=tg; MAX=*Btrue; MIN=*Gtrue; rmax=rb+delayB; rmin=rg-delayB;}
  else {s=tg/tb; tmin=tb; MIN=*Btrue; MAX=*Gtrue; rmin=rb+delayB; rmax=rg-delayB;}

  X= exp(-MAX*rmin) + exp(-MIN*rmax) -1
     - MAX*MIN  *( rmax*rmax/2      *(1-tmin*( MAX-d(s-1)*MIN) )
                  +rmin*rmin/2      *(d(s-1)+tmin*(MAX-d(s-1)*MIN) )
                  -rmax*rmax*rmax/6 *(MIN*(1+d(s-1))-MAX)
                  -rmin*rmin*rmin/6 *(MAX*2*d(s-1) +MIN*(d(s-2)-d(s-1)) )  );
  Y=1 - tmin/2      *( MAX+ (2-d(s-1))*MIN )
    + tmin*tmin/6   *( MAX*MAX + MAX*MIN*(6-2*d(s-1)) + MIN*MIN*(6-d(s-2)-5*d(s-1)) )
    - tmin*tmin*tmin/24 *( MAX*MAX*MAX
                          +MAX*MAX*MIN *(14-3*d(s-1))
                          +MAX*MIN*MIN *(36-2*d(s-2)-25*d(s-1))
                          +MIN*MIN*MIN *(12-d(s-3)-6*d(s-2)-11*d(s-1)) );
  *Ctrue=(Cobs-(rb+rg)*Bobs*Gobs)/(pb*pg*X + Cobs*tmin*Y);
  }
```

-(DOS)--- CI Smith approx - simple.c Top L1 (C/*l Abbrev)

Computer codes

- Codes that implement CIS correction (Fortran, C)

CI Smith approx - in & out of channel.for - GNU Emacs at hos64748

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```
      SUBROUTINE DSMITHAPPRINOUT(B,G,C,GCH,CCH,
     1                 TB,TG,HB,HG,HBCH,HGCH,  RB,RG,L12,RGCH,L12CH)

C      Implements COX-ISHAM/SMITH solution including OUT_OF_CHANNEL gammas
C       and coincidences.
C       ie  implements ICRU 52 equations 5.80 with 5.51.
C
C      The approximation is valid for any deadtime & resolving time values,
C          subject to the usual limitations :
C           - the safe limitation is  (HB+HG) < minimum(TB,TG)
C           - the safe limitation is  (HBCH+HGCH) < minimum(TB,TG)
C           - resolving times must be set to properly cover the real
C                time-distribution for each coincidence unit
C                (for details see ICRU 52).
C
C      Written in FORTRAN
C
C      Require to measure in-channel gammas and coincidences, and also
C      the total gammas and coincidences.     The total refers to both
C      in- and out- of channel events.
C
C      Usually have 5 counters : B, G, C, GCH, CCH
C      and two coincidence AND gates.   One AND is for the total gammas
C      and total coincidences (ie both in- and out-of channel).   A
C      second AND is for in-channel gammas and in-channel coincidences.
C

C     INPUT
C       B        observed beta count rate
C       G,C      observed TOTAL gammas (in whole gamma spectrum) and
C                         their coincidences with the betas
C       GCH,CCH  observed IN-CHANNEL gammas, and their coincidences
C                         with the betas.
C       TB,TG      DEAD TIMES
C       HB,HG      EFFECTIVE RESOLVING TIMES FOR B AND G IN
C                                      THE MEASUREMENT OF C
C       HBCH,HGCH  EFFECTIVE RESOLVING TIMES FOR B AND GCH IN
C                                      THE MEASUREMENT OF CCH.
C
C     OUTPUT
C       RB      CORRECTED B
C       RG      CORRECTED G
C       L12     CORRECTED C (APPROX)
C       RGCH    CORRECTED GCH
C       L12CH   CORRECTED CCH (APPROX)
C
```

-(DOS)--- **CI Smith approx - in & out of channel.for** Top L1 (Fortran)

$\tau_\beta = s \cdot \tau_\gamma$, s non-integer,

out-of-channel events

Summary

- Work with $\tau_\beta = n^*\tau_\gamma$ or $\tau_\gamma = n^*\tau_\beta$, n integer > 1, or near there.

- Avoid using τ_β/τ_γ or $\tau_\gamma/\tau_\beta \in [0.5, 1.9]$.

- If you intend using approximate and not exact CIS correction, eschew standardising too high countrate sources (< 100 kcps).

The End

CIS coincidence correction

- Particular case $r_\beta = r_\gamma$ and $\tau_\beta = \tau_\gamma$

Laplace transforms of the diff. equations

$$\int_0^\infty e^{-st} f(t)dt = F(s) \qquad \int_0^\infty e^{-st} f'(t)dt = sF(s) - f(0)$$

$$\begin{cases} sQ_\beta(s) - q_\beta(0) = -Q_\beta(s)\varrho_\gamma + e^{-s\tau}Q_\gamma(s)\varrho_\beta \\ sQ_\gamma(s) - q_\gamma(0) = -Q_\gamma(s)\varrho_\beta + e^{-s\tau}Q_\beta(s)\varrho_\gamma \end{cases}$$

$$\begin{cases} (s + \varrho_\gamma)Q_\beta(s) = q_\beta(0) + e^{-s\tau}Q_\gamma(s)\varrho_\beta \\ (s + \varrho_\beta)Q_\gamma(s) = q_\gamma(0) + e^{-s\tau}Q_\beta(s)\varrho_\gamma \end{cases}$$